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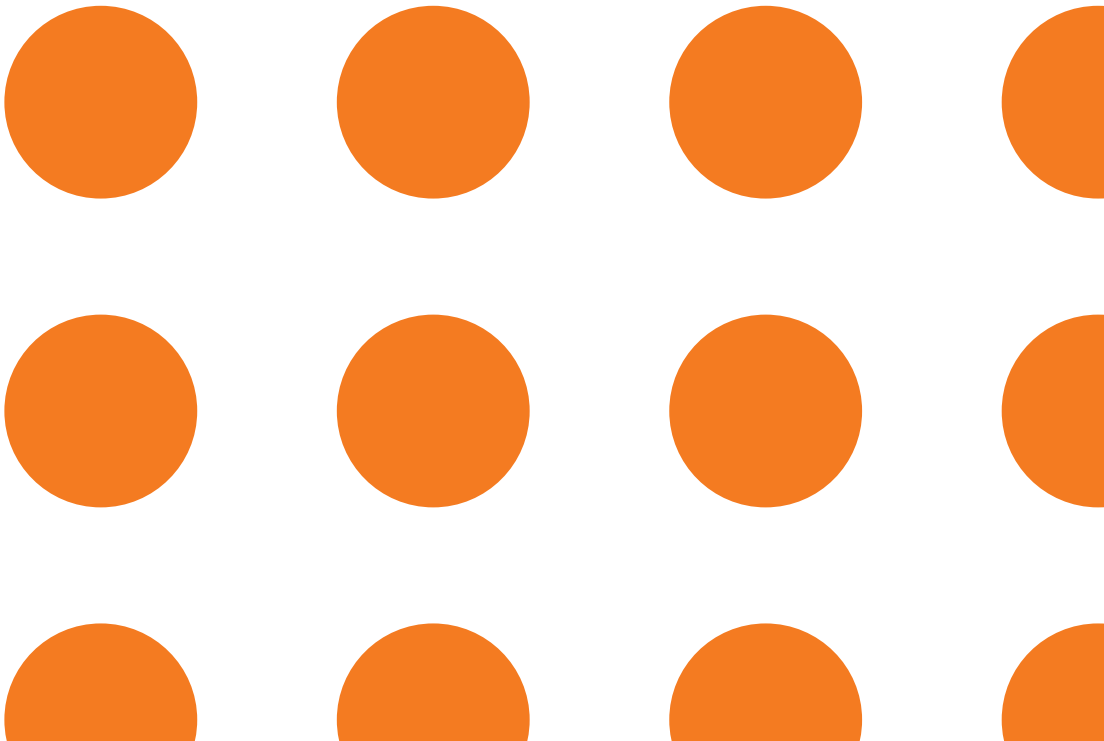
MÁRIA MINÁROVÁ

RHEOLOGY
Fundamentals, tools, examples



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2025

Since rheology is quite a young science, there are still a lot of issues to be studied, observed, explored. I hope this book will inspire and stimulate engineers, material scientists, chemists, mathematicians or their teams to find out a beauty of rheology, possibilities of its exploitation and excitement of the rheological investigation. Built up upon fundamentals, by using existing tools, or the tools introduced in the book, four themes are explored in more detail: 1. biological material – human plantar aponeurosis tissue under creep test and relaxation test; 2. viscoelastic structure subject to a dynamical load; 3. viscoelastoplastic model of concrete; 4. dissipation of energy as one of the decisive factor justifying or not, the linear approach in subsequent investigation. Conditional stiffness, Duhamel hereditary integrals, time dependent moduli, Prony series, variation inequalities, etc., are used as tools herein. Everything is built up on fundamentals, thermodynamic consistency is required and examined.

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Mojim drahým rodičom

Devoted to my dear parents

Contents

Preface

"Fundamentals, tools, examples" - the sub-title of this book discloses the base of its mission. All investigation is built upon existing explored results. In the book, the reader can find rheological and viscoelastic preliminaries, together with the smart tools developed a long time ago or more recently. Although the book is not intended to bring a complete state-of-the-art of rheology, I hope that some interesting examples of rheological investigation together with the applications of the tools will be beneficial for the reader as well. The considerations proceed from simple to complex; it is synoptic, and easy to read, raising new inspiring queries.

The book is destined first of all for engineers, Ph.D. students or undergraduate students - future engineers; material scientists, chemists, and civil, mechanical or biomechanical engineers. The only prerequisite for the beneficial reading is a basic knowledge of integral calculus and the theory of linear ordinary differential equations. All investigations of the author in the field stand on the preliminaries. Appropriate mathematical tools are utilized for particular problems.

Alike over all rheology, the main issue focused in this book is devoted to basic considerations about the constitutive relations, i.e. the relations between the stress and the strain, of the so-called soft materials or non-Newtonian fluids – those matters which are neither purely elastic nor purely viscous.

As it is usual in mechanical investigating, at its very beginning there are some basic thermodynamic assumptions that should be taken into account. The ensuring of the linearity is focused if possible, together with the validity of the Boltzmann principle of superposition. Rheological structural forms and pictograms of rheological models support the synopsis alongside all considerations. Duhamel convolution integrals, (hereditary integrals) are employed to perform the explicit dependence of stress on strain or vice versa. The quantification of dissipative energy is provided as an important tool for detecting whether the linear approach is still of satisfactory exactness. Otherwise, the released (dissipated) heat causes the essential temperature increase that non-negligibly influences some of the

physical parameters and the non-linear approach has to be employed. More complex models of the regular or irregular structures are treated and set up as models of some matters such as tissues, building materials, soil, etc.
Moreover, the three-dimensional viscoelastic investigation is provided, along with a short theoretical background and the application on a viscoelastic body subjected to a periodic dynamic load.
Among other examples the human plantar aponeurosis is focused on and explored in more detail.
A brief overview of viscoelastoplasticity is provided as well, including its mathematical background based on the variational inequalities serving as a tool for coping with a singularity present there. The viscoelastoplastic study of a simplified model of overloaded concrete is appended, performing how the tools work.
The investigation presented in the book was supported by grants VEGA 1/0036/23, VEGA 1/0155/23 and KEGA 030STU-4/2023.

Introduction

Rheology studies the deformation and the flow of the matter. It deals with semi-solids or semi-liquids, i.e. materials with mechanical properties and mechanical behaviour are situated somewhere between purely elastic matters (Hookean elastic matters) and purely viscous (Newtonian) fluids. We say that such materials exhibit a rheological behaviour. Many soft solid materials and elastic-viscous liquids can be named, which is the focus of rheology: metals, polymers, glass, concrete, soil, asphalt mixtures, suspensions, gels, pastes, solid biomaterials, living body liquids such as blood, lymph, lubricants, dilute solutions of polymers and many other materials. These materials are characterized by a flow strongly influenced by their viscosity. Sometimes the viscosity itself undergoes a fundamental change. Thist can happen due to changes in physical conditions or significant ambient properties, e.g. temperature, pressure, electrical potential, etc. We can take the glass as a first example of the specific mechanical behaviour depending on the temperature. The glass starts to flow, and it even undergoes the phase transition at a sufficiently high temperature and becomes solid again after the temperature decreases. As it is experienced and warned by the glass industry, the melted glass mass has to be cooled down very slowly, usually in an oven, to avoid becoming too brittle after the backward solidification. It is similar for metals and many other materials. However, some materials, e.g. eggs, solidify under higher temperatures. On the other hand, there are other influences which can change the rheological properties of the materials. The viscosity of ketchup increases when shaken, sugar caramelizes after being melted, soil consolidates under pressure, etc.

Rheology finds its application in various fields of life. First of all the material industry has to be mentioned, the polymers and polymer-based materials are of interest, the newly developed building materials are being explored; the food industry uses many soft materials, pastes, foams, etc. Each time the theoretical investigation together with laboratory test enables us to predict the material quality and suitability for its future usage.

The theoretical rheological investigation starts with building up

a rheological model. The model should match the material as precisely as possible, concerning the supposed future usage of the material and the expected load (type and magnitude). Some simplification is always present in such a procedure. Most rheological models simulate the materials or devices (see the viscoelastic damping isolator in Figure 1) within given time intervals of loading, with other physical influences like the temperature or the pressure gradient being either neglected or not. When the appropriate rheological model is built up, the theoretical investigation simulation of various load impacts under various physical situations can be carried out, and the prediction of the material mechanical behaviour is done.
The book is organized as follows: In the first chapter, we briefly focus on thermodynamics. Thermodynamics, its laws, the dissipation of energy phenomenon, and the conservation laws are the essential assumptions that need to be fulfilled before the rheological investigation. Next chapter refers to rheology and viscoelasticity. A rheological model itself is described, and the fundamental elements, the parallel and the serial connection, are introduced. The standard tests and the mechanical constitutive relations are of interest. The time-dependent material characteristics, i.e. the relaxation modulus and the creep compliance modulus are set up, followed by the conditional stiffness differential operator. Furthermore, multi-element viscoelastic modelling comes into account. Afterwards, multidimensional viscoelasticity is aimed. Lastly, the plastic element is incorporated in a rheological model, and the rheological modelling beyond the viscoelasticity theory is provided. All investigations are documented by the corresponding model elaboration attached to each particular issue. Some practical utilization and applications are also added.

A little bit of history

Rheology is quite a new discipline. Its official origin is dated in 1929, when the first rheological meeting took place in Washington, and when the official definition of rheology was established. However, the roots of rheology can be found. And, much earlier,

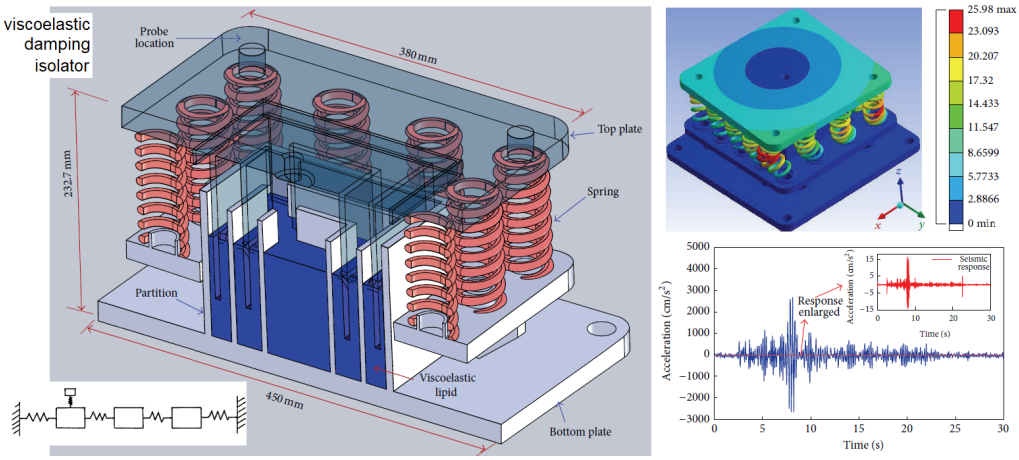

Figure 1: Viscoelastic damping isolator [16].

even in antiquity. The most popular ancient rheological statements are "Παντα ρει" ("Panta rei" - "Everything flows"), told by the Greek philosopher Heraclitus in the 6^{th} century B.C, and "The mountains flow before the Lord", told by Jewish prophetess Deborah around the 12^{th} century B.C. Even her name, Deborah, served as an inspiration for the naming of the non-dimensional number N_D which quantifies the ability of materials to flow. $N_D \in [0, 1]$ evaluates the ratio of the time of relaxation to the time of observation. Deborah knew that the mountains flow, as everything flows. But the mountains flow before the Lord's sight and not before the man's one. It is so because the man's life is too short to observe the folding deformation - the flux of the mountains which takes centuries or millennia, while the time of God's observation is infinite. Then, the marginal values of Deborah number 1 and 0 are respectively attached to pure liquids and pure solids [51].

In 1678, Robert Hooke introduced the linear law expressing the relationship between stress and strain; a couple of years later, in 1687 Isaac Newton introduced his study of the steady shear flow in fluids. In his Principia (Philosophiae Naturalis Principia Mathematica) we can find his famous hypothesis: "The resistance which arises from the "lack of slipperiness" of the parts of the liquid, other things being equal, is proportional to the ve-

locity with which the parts of the liquid are separated from one another". This lack of slipperiness is called the viscosity nowadays. Both laws are widely used in rheology up to now, the pure solid-like behaviour and the pure fluid being considered just as "boundaries". Indeed, every usual materials' mechanical behaviour stands somewhere between these two extremes. Later on, scientists like James Clerk Maxwell, Lord Kelvin and Woldemar Voigt began to use the term "elastic-viscous liquid" or "elastic liquid" in the second half of the 19^{th} century [51].
The first factual rheological (viscoelastic) test is dated 1835, when Wilhelm Weber imposed a tension load on the silk threads. An initial immediate extension of the magnitude Δl appeared first as expected, but despite the load remaining unchanged, the prolongation of thread continued, though with a decreasing rate. After the thread prolongation had stopped, the stress was removed instantaneously. By the expectation, an immediate shortening of the magnitude Δl occurred, and then the thread continued its shortening more and more slowly, towards the original length. After a sufficient period of time, the total recovery was reached again.
Summarizing the previous investigations in the field, Boltzmann at the very end of the 19^{th} century rearranged the differential form of constitutive relation into integral (integro-differential) form, even generalized for three-dimensions.
Boltzmann is also famous for his principle of superposition: "The value of a characteristic function of a system is equal to the sum of all changes induced in the system by the driving functions which have been applied to it throughout its history" [54]. The superposition principle is widely used along all investigation described in this book. At the beginning of the 20^{th} century, John Henry Poynting and Joseph John Thomson started to use rheological elements and proposed their graphical representation - pictograms: the spring as a representative of Hookean elastic behaviour and a dash-pot as a representative of a Newtonian flow within the rheological investigation. The Prony series was first used by Richard Schapery and Sunwoo Park in 1999 in the attempt of the fit the experimental data with the theoretical investigation. As it is possible to gain the Prony coefficients from the experimental data, they are now commonly used in the sci-

entific community dealing with amorphous polymers. Both the Boltzmann superposition principle and the regularization were exploited in the system of linear equations. Afterwards, the recurrent expression of the Prony series form was used.

Chapter 1

THERMODYNAMICS

1.1 Principles of phenomenology

Alongside our investigation, the *general theory for the phenomenological construction of the constitutive equations* of non-living matters is employed, following the physical principles of the thermodynamic phenomenology, [2, 9]:

- Determinism principle - the status of the body at time t is determined by its history (stochasticity is not involved)
- Causality principle - the cause evokes an effect,
- Smooth ambient - singularities are regarded just theoretically,
- Memory (smooth, indifferent, degenerated, etc.) is/is not taken into account - time invariant and time variant problems are observed where ageing of matter comes into account,
- Proper representation - the model has to match with the represented phenomenon,
- Coordinate system invariance - results released from the model do not depend on the chosen coordinate system.

1.2 Thermodynamic laws

It is a well-known fact that thermodynamics operates macroscopically. It means neither a single particle (atom or molecule) of solid, liquid or gas is traced, nor an exact motion of the particles, nor the mutual interaction and the collisions are of interest, since their influence is negligible, and the focus is just on a global impact of this motion and an interaction with the entire system. Within the large-scale consideration, the macroscopic properties of the system are observed. First of all, it is volume, pressure and temperature; then the related parameters such as energy, density, mechanical potential, etc.
Thermodynamics has to be taken into account in an investigation of many processes of natural or engineering sciences. These processes should be thermodynamically consistent, i.e. they should be in no contradiction with any law of thermodynamics. All rheological investigations are done in this thermodynamic background. Thorough, the thermodynamic laws are valid for closed or isolated systems. Indeed, sometimes it is not an easy task to ensure thermodynamic consistency. Especially in the case of biological structures dealt with by rheology, e.g. the human, animal, or herbal tissues that can hardly exist apart or separated from the organism. That is why these systems can be regarded neither isolated nor close. Quite the opposite. They are open systems, which means they interchange energy and matter with their surroundings.
Although in reality, the thermodynamic laws are valid for either isolated or closed systems only, necessarily, in the short-term observation, they are regarded either as isolated or closed ones [2, 18], and the laws of thermodynamics are used after all.
Let us briefly state the laws of thermodynamics:

1. Inside an isolated system, the energy cannot be created or destroyed

$$\Delta\left(U_k + U_p\right) = \Delta\left(W + \sum_{\alpha} H_{\alpha}, \right), \tag{1.1}$$

where $\Delta\left(U_k + U_p\right)$ is the increment of the sum of the kinetic energy U_k and the inner potential energy U_p, $\Delta\left(W + \sum_{\alpha} H_{\alpha}\right)$

is the increment of the sum of work W of the outer forces per time unit and H_α represents other types of non-mechanical energy added, supplied heat is included herein - all issues quantified per volume unit and time unit $\left[\frac{J}{m^3 s}\right]$.
If all energy functions in (1.1) are continuously differentiable, the differentiated form of the law is arisen:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(U_k + U_p\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(W + \sum_\alpha H_\alpha\right). \tag{1.2}$$

This law is known as the conservation of energy.

2. Heat can never be transformed completely into a work-performing type of energy. Or (equivalent expression of the law): Heat does never flow on its own from the colder place to the warmer one unless a forcing energy is expended, [9]:

$$\Delta S \geq \frac{1}{T}\Delta Q \tag{1.3}$$

in the discrete case and

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geq \frac{1}{T}\frac{\mathrm{d}Q}{\mathrm{d}t} \tag{1.4}$$

if S and Q are continuously differentiable.
Here $S\left[\frac{J}{K}\right]$ is the global entropy of the body, Q $[J]$ the heat and T $[K]$ the temperature.
In the case of no loss of the mechanical energy, a completely *reversible process* is concerned and the equality in (1.3) or (1.4) characterize the process

$$\Delta S = \frac{1}{T}\Delta Q, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{T}\frac{\mathrm{d}Q}{\mathrm{d}t}.$$

Finally, for an *irreversible process*, energy dissipates, so the strict inequality in (1.3) and (1.4) comes into account:

$$\Delta S > \frac{1}{T}\Delta Q, \quad \frac{\mathrm{d}S}{\mathrm{d}t} > \frac{1}{T}\frac{\mathrm{d}Q}{\mathrm{d}t}.$$

The equivalent formulation says that the entropy production is always non-negative:

$$\left(\frac{\mathrm{d}S}{\mathrm{d}t}\right)_V = \frac{\mathrm{d}S}{\mathrm{d}t} - \int_{A(t)} \frac{q_i n_i}{T} \mathrm{d}A - \int_{V(t)} \frac{\rho h_c}{T} \mathrm{d}V \geq 0,$$

where $\left(\frac{\mathrm{d}S}{\mathrm{d}t}\right)_V$ the inner entropy production, n_i the component of the unit normal vector to the boundary, $q_i \left[\frac{W}{m^2}\right]$ is the heat flow density vector component, $h_c \left[\frac{J}{s.kg}\right]$ is the flux coefficient of the non-mechanical energy in mass unit per time.

3. The entropy magnitude of a system approaches a constant value alongside the temperature approaching absolute zero.
   The equivalent formulation says that it is impossible in reality to reach absolute zero temperature.

In our considerations, the first law of thermodynamics will be mostly referred to.

Chapter 2

RHEOLOGY

Alongside continuum mechanics, the mechanical response of a body or a system to a load is focused. It means that the relation between strain ε and stress σ, called *constitutive relation* or *physical relation* is always essential. Within linear elasticity theory, the proportional relation between stress and strain can be expressed explicitly; either a stiffness or a compliance tensor is utilized as the material properties parameter. Similarly, in the linear fluid flow theory, the proportional relation exists between the stress and the strain rate. Indeed, in more complex tasks of continuum mechanics, these relations either are not linear, since e.g. the physical parameters become dependent on an unknown function, or an implicit expression is provided while the explicit forms need additional entries. The time variable and its derivatives are widely utilized. Moreover, a response difference during the loading and unloading process called *hysteresis*, [20, 8, 42] occurs. Let us recall that the hysteresis occurs when the same input yields a different output during loading and unloading process, i.e. the stress - strain curve during loading differs from that during unloading, see Figure 2.3. Rheology studies these relations, where different types of material response (elastic, viscous, plastic, etc.) act together in synergy.

Stress and strain tensors

Stress and strain are always essential issues in continuum mechanics. They are tied up through a modulus involving the physical properties of the material. They are referred to alongside all investigations presented in this book.
Stress is defined as an external deforming force per an area. Generally, it is a tensor of 2^{nd}-order, a matrix with its nine components for three dimensions. However, in equilibrium stage, in order to ensure the balance of angular momentum, the shear stresses on two perpendicular planes of a differential element are equal. From this, the symmetry of stress tensor $\sigma_{ij} = \sigma_{ji}$ follows. Consequently, we have to remember six instead of nine values of the tensor and for the sake of future better handling we rearrange these six values in a vector:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix}. \tag{2.1}$$

Let us emphasize moreover the subscript convention herein: The first of two subscripts of stress tensor $\boldsymbol{\sigma} = \sigma_{ij}$ indicates the normal of the plane element where the stress (force) is applied, the second subscript indicates the stress component direction.
Similarly $\boldsymbol{\varepsilon}$ - the strain tensor of the 2^{nd}-order and its symmetry, i.e.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{ji},$$

in sense of displacement u, enables us to use it in the form of a vector:

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix} \rightarrow \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix} = \boldsymbol{\varepsilon}. \tag{2.2}$$

Rheological material

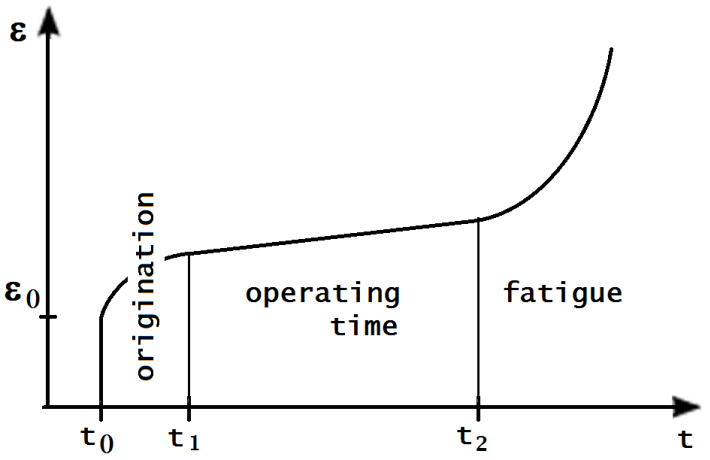

Figure 2.1: Creep of a material within its life span. Three phases of the mechanical behaviour of a material - initial premature phase, operating time, and failure, [1].

Many materials of ordinary usage regarded as *elastic* undergo quite turbulent changes during their life span. Mostly, a strong non-linearity accompanies the origin of the material. Later on, the non-linear behaviour usually fades with the material maturing. Only after the maturing, its typical elastic behaviour arrives and stays with the material until fatigue appears due to overloading, depreciation or other reasons for degradation. Then, a non-linear decay happens, resulting in fatigue and the failure of the material. An example of material behaviour is provided in Figure 2.1, therein, subjected to constant stress maintained during its whole life span. Various material behaviour is depicted in Figure 2.2. The mechanical behaviour of a material with time lapsing can be performed as a split into three phases:
- **primary phase** - $t \in \langle t_0, t_1)$, origin with a non-linear behaviour with a high initial deformation rate due to a chemical process, a temperature or pressure activation while being formed; then the deformation rate decreases as the material matures.
- **secondary phase** - $t \in \langle t_1, t_2\rangle$, operating life span of the material, when no or just slow and linear deformation of a very low

slope occurs.
- **tertiary phase** - $t > t_2$, fatigue of the material culminating in its damage.
The elastic modulus of matured elastic material remains constant or almost constant during the operating time.

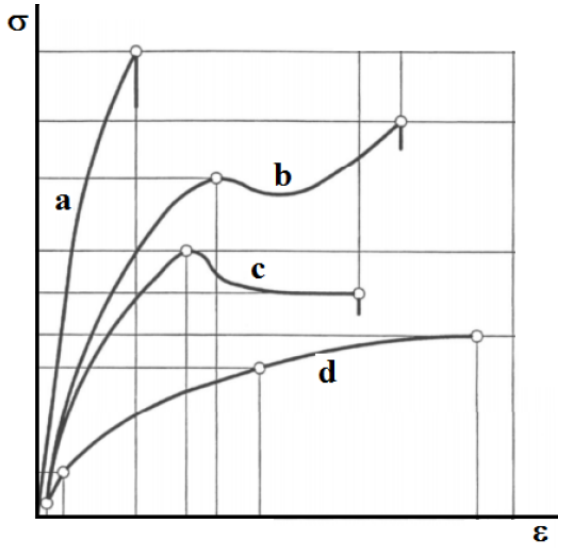

Figure 2.2: Stress-strain graphs of materials of characteristic types, a) - brittle material, b) and c) - tough materials with a yield point, d) - tough materials without any yield point, [42].

2.1 Rheology and viscoelasticity within

2.1.1 Rheological model and its properties

Rheological model of a material (matter) is a model standing behind a material as far as mechanical behaviour is concerned. Such model simulates the mechanical reaction of the material to an action. In this book, only a static or quasi-static mechanical load is considered an action; accordingly, no vibrations are induced. Then, under the imposed load, the reaction, i.e. the mechanical response, is studied. However, rheological material

response is often determined by other physical circumstances, e.g. a temperature in the ambient or pressure difference, a chemical or electrical potential, etc. A rheological model always consists of several (at least two) fundamental elements connected in various configurations. The aim is always to fit the real material mechanical behaviour as precisely as required.
Configuration or constellation means that all particular submodels and fundamental rheological elements are connected mutually, either *in parallel* or *in series*, creating the final geometry. Indeed, every fundamental element contributes to the entire mechanical behaviour, influencing the global constitutive relation. Hence, both the configuration and the physical relations of the particular members predestine the global constitutive relation. The global constitutive relation further theoretically determines the mechanical behaviour of the rheological model.
Generally, the current rheological material response is influenced not only by the current load magnitude, its duration, and velocity, but also by the so-called *history of the load*. For this reason, an additional time variable reflecting the history of action is incorporated into the rheological models.
Since all the observed processes are embedded within the range of values where those influences are negligible, all considerations in this book are done without the temperature and pressure influences. The processes are regarded as isothermal and isobaric. Under these assumptions, the formal definition of the rheological model can be designated as follows.

Definition 1
Let us suppose that the stress function $\boldsymbol{\sigma}$ and the strain function $\boldsymbol{\varepsilon}$ are symmetric tensor functions.
A system that consists of

1. physical (constitutive) relation between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$,
2. potential energy $U_p \geq 0$,

is called the *rheological model*.

As it is habitual in continuum mechanics, the thermodynamic consistency of a rheological model is always required:

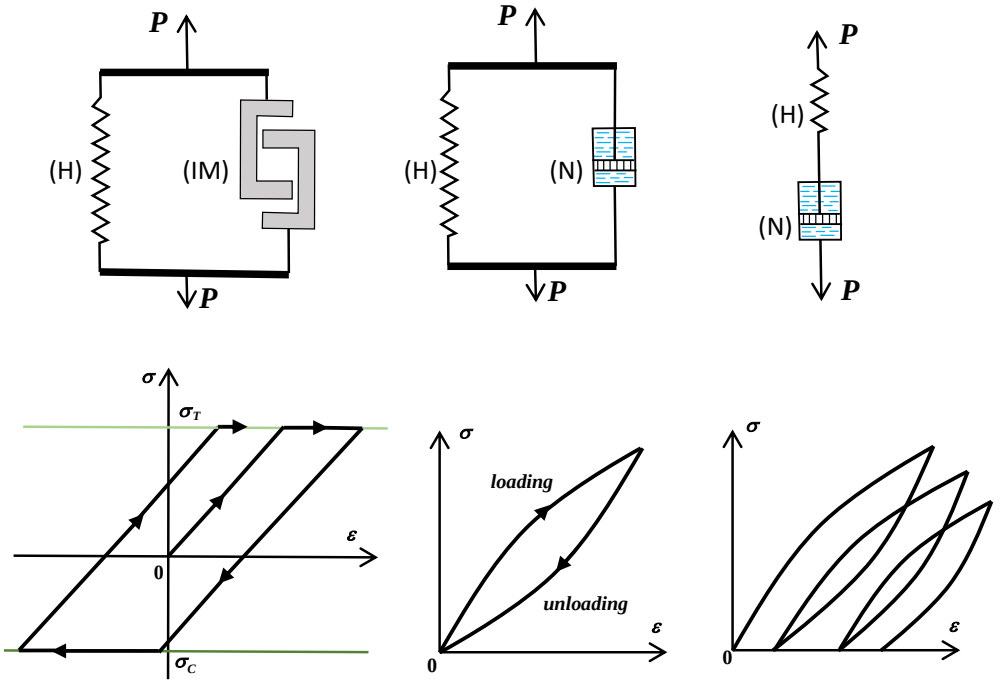

Figure 2.3: Examples of hysteresis of rheological models subjected to tension stress; $(H)|(IM)$, $(H)|(N)$, $(N1)-(N2)$.

Definition 2
The uni-axial rheological model is thermodynamically consistent if the quantity called the dissipation rate

$$\dot{U}_d = \langle \dot{\boldsymbol{\varepsilon}}, \boldsymbol{\sigma} \rangle - \dot{U}_p \tag{2.3}$$

is non-negative in the sense of distributions for all $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$, [20]. Herein, $\langle \dot{\boldsymbol{\varepsilon}}, \boldsymbol{\sigma} \rangle$ is the scalar product of strain rate and stress, U_d stands as the amount of dissipated energy and U_p for the potential (stored) energy.
Nevertheless, in Chapter 3.3, quantification of dissipated energy is provided, which can afflict the linear conception of the investigation. Namely, the dissipated energy is mostly converted to heat that spreads to the ambient, resulting in a temperature increase that can further affect the physical parameters of the objects. In such a case, the linear approach of exploring is no longer relevant.

Remark

1. Unless declared otherwise, verbatim, alongside the book, the static or quasi-static load is taken into account when referred to as a "load". In doing so, an entity subjected to the static or quasi-static load is deformed, but the load causes no vibrations.

2. The dot above a variable stands for its time derivative $\dot{\varepsilon} = \frac{\mathrm{d}\varepsilon}{\mathrm{d}t}$; analogously double dot stands for its second time derivative $\ddot{\varepsilon} = \frac{\mathrm{d}^2\varepsilon}{\mathrm{d}t^2}$, and so forth.

3. The expressions *constitutive relation, stress-strain relation,* $\boldsymbol{\sigma \sim \varepsilon}$ *dependence, physical relations* are used equivalently.

2.1.2 Fundamental rheological elements and their physical properties

Various fundamental rheological elements can be involved in a rheological model. As far as the mechanical behaviour is concerned, each of them represents a specific physical feature. The mechanical behaviour of each element is then governed by its specific stress-strain (physical) relation. The pictograms of particular rheological elements are designed in a way that the reader can anticipate their behaviour under a mechanical load. Representative abbreviations of fundamental elements (H), (N), (StV), etc. for Hookean elastic, Newton viscous and Saint-Venant plastic elements (matters) are of use when creating structural formulas of particular rheological model, e.g. $(H)|(N)$, $(StV)|[(H)-(N)]$, see the configuration in Chapter 2.1.3.
Moreover, pictograms of rheological models are used for a better synopsis involving the pictograms of fundamental elements in an appropriate configuration, Figure 2.4. Accordingly, the Hookean elastic element (H) is graphically substituted by a spring, the Newton's viscous (N) matter by a dashpot, the Saint-Venant plastic element (StV) by two plates superimposed on each other, friction force supposed on their common area acting against the applied outer force, etc. Indeed, other rheological elements can

take part in the rheological models as well, performing a more general rheological behaviour. These elements are depicted in Figure 2.4 together with their graphical symbols and their abbreviations. In this book, just three fundamental elements are used and studied: (*H*), (*N*) and (*StV*), and various models consisting of these elements. In Figure 2.8 the performance of the

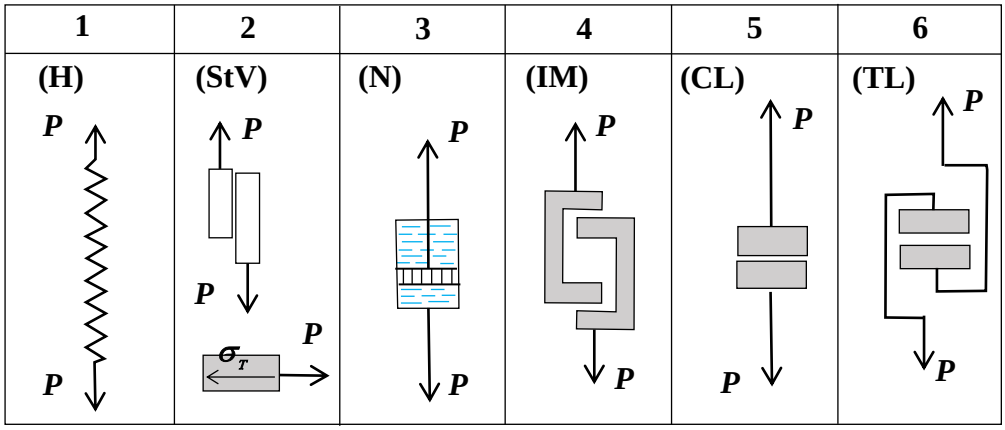

Figure 2.4: Elementary rheological elements, abbreviations and pictograms: 1) Hookean elastic matter, 2) Saint-Venant's plastic matter, 3) Newton's viscous matter, 4) Indurate matter, 5) Compressive ligature 6) Tensile ligature .

related graphs of the stress-strain relations of elastic, viscous and plastic elements are provided. For the sake of better apprehension, the pictograms involve arrows denoted by *P*, representing tension force. Of course, in the case of compression force, the arrows will be of the opposite orientation. Herein *P*, tension or compression is imposed, aligned with the degree of the freedom direction. Habitually, the tension force has a positive sign, and the pressure force a has negative sign.
Let us have a look at our three elements and their exertion.

Elastic element (*H*)

In figures, usually a spring is used as a pictogram of an elastic element. The element is concerned *purely elastic*. Hence,

the mechanical behaviour is governed by Hook's law, a linear stress-strain relation. In tensor form, [7]:

$$\boldsymbol{\sigma} = \mathbf{A}\boldsymbol{\varepsilon} \tag{2.4}$$

It is worth saying that equation (2.4) represents a proportional relation between 2^{nd}-order tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, $\mathbf{A}$ being the proportional factor, the 4^{th}-order material tensor, modulus of elasticity.
Indeed, in the case of a tensor to vector alignment in the sense of (2.1) and (2.2) of both $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, $\mathbf{A}$ reverts to a matrix. This matrix is called the anisotropic body tensor. Generally, $\mathbf{A}$ involves both volumetric and deviatoric change, see Figure 2.5; and both phenomena can be regarded separately in the case of an isotropic material. For this sake, it is worth writing the matrix $\mathbf{A}$ as the sum of volumetric and deviatoric matrices $\mathbf{A}_{vol}$ and $\mathbf{A}_{dev}$, respectively. This matrix can be further split into four 3x3 submatrices as in (2.6).

$$\mathbf{A} = \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Y} \end{pmatrix} = \mathbf{A}_{dev} + \mathbf{A}_{vol} \tag{2.5}$$

with $\mathbf{Z}$ being a zero 3x3 matrix,

$$\mathbf{A}_{dev} = \begin{pmatrix} \mathbf{Z} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Y} \end{pmatrix}, \quad \mathbf{A}_{vol} = \begin{pmatrix} \mathbf{X} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{Z} \end{pmatrix}, \tag{2.6}$$

$$\mathbf{X} = \begin{pmatrix} 1+2\mu & \lambda & \lambda \\ \lambda & 1+2\mu & \lambda \\ \lambda & \lambda & 1+2\mu \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} 2\mu & 0 & 0 \\ 0 & 2\mu & 0 \\ 0 & 0 & 2\mu \end{pmatrix},$$

where $\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$ are the Lamé coefficients involving the Young modulus of elasticity E and the Poison ratio number ν.
Due to zero potential energy dissipation of the elastic element (H) standing either solely or connected *in series*, the deformation of (H) is completely reversible. No energy is dissipated during the loading, which means that all potential energy U_p acquired during the loading process is kept and stays ready to be utilized for complete recovery as soon as the load is removed.

In the one-dimensional viscoelastic modelling, we use the one dimensional form of the Hook's law

$$\sigma = E\varepsilon \tag{2.7}$$

The potential energy is equal to the energy of deformation herein. Quantified per volume unit, for continuous $\varepsilon(t)$ it can be written in the form of the Stieltjes integral:

$$U_p(t) = \int_{\varepsilon(0)}^{\varepsilon(t)} E\varepsilon(\theta)\mathrm{d}\varepsilon(\theta) = \frac{1}{2}E\Big(\varepsilon^2(t) - \varepsilon^2(0)\Big). \tag{2.8}$$

Expressed in the form of the Riemann integral which will be of further use in Chapter 3, we have the integral over time

$$U_p(t) = \int_0^t \dot{\varepsilon}(\theta)\sigma(\theta)\mathrm{d}\theta. \tag{2.9}$$

All energy added to an elastic member during the loading process is stored, and no energy is dissipated, which documents the thermodynamic consistency (2.3) of the elastic member straightforwardly, $U_d = 0 \implies \dot{U}_d = 0$. The total recovery of the elastic member arrives by using this potential energy immediately after the load is given away.
It is interesting to add herein that since the dissipated energy is equal to zero, both the loading and unloading processes are equal (without a damping or delay), accordingly no hysteresis is present, see Figure 2.7, left.

Viscous element (N)

In the figures, the dashpot is usually used as a pictogram of a viscous element. A Newtonian liquid is thought herein; the proportionality between the stress and the first time derivative of strain function is active. In other words, the physical process is governed by Newton's law for liquids - the linear relation between the stress and the strain rate. Besides, the deviatoric and volumetric components can be expressed and treated separately in three dimensions:

$$\begin{aligned} \boldsymbol{\sigma}_{dev} &= \boldsymbol{\eta}\dot{\boldsymbol{\varepsilon}}_{dev}, \\ \boldsymbol{\sigma}_{vol} &= \boldsymbol{\zeta}\dot{\boldsymbol{\varepsilon}}_{vol}, \end{aligned} \tag{2.10}$$

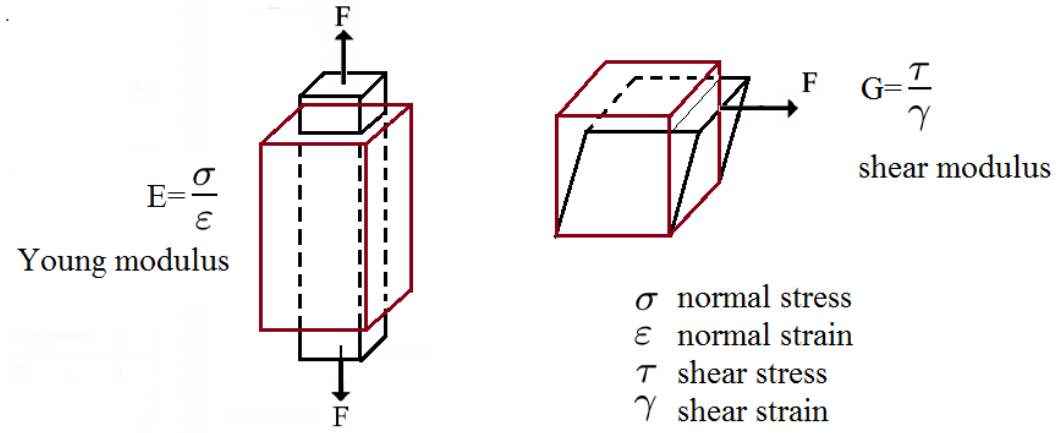

Figure 2.5: Young modulus of elasticity and shear modulus.

where $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ are deviatoric and volumetric proportional material tensors, respectively. When, like in our case, incompressible liquids are considered, i.e. the volume remains constant, only the deviatoric component takes part in the stress-strain relation, Figure 2.6. Hence in the uni-axial case, the constitutive relation of incompressible liquid can be expressed in the form of a single scalar equation:

$$\sigma = \eta\dot{\varepsilon}. \tag{2.11}$$

Here, η $[Pa.s]$ is the dynamical viscosity.

Remark
1. The kinematic viscosity ζ $[m^2s^{-1}]$ is also of use in fluid flow theory. The relation between these two (kinematic and dynamic) viscosities is $\eta = \zeta.\rho$ where ρ $[kgm^{-3}]$ is the density of the liquid.
2. The term *viscosity* is utilized in fluid mechanics. Newton's law of viscosity stands as governing stress-strain (physical) equation quantifying the proportionality between the stress and the strain rate. Newton's law of viscosity states that under unchanged temperature, the shear stress between adjacent fluid layers is proportional to the shear deformation rate between

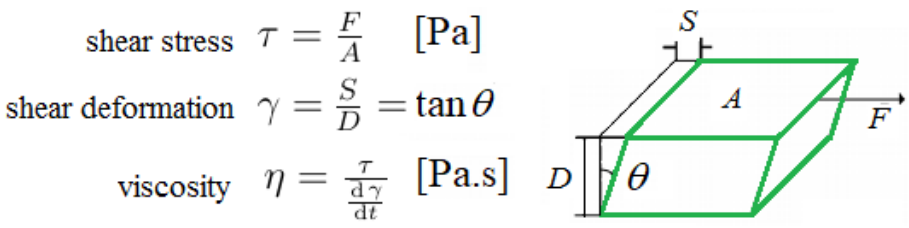

Figure 2.6: Shear viscosity of a liquid.

these two layers.

$$\tau = \eta\dot{\gamma},$$

where τ $[Pa]$ is the shear stress of viscous liquid and γ $[-]$ is the shear deformation. Nevertheless, since we need to combine the elastic and the flow deformation in viscoelastic exploration, we will do a unification of the notation herein, using σ for the shear stress instead of τ and ε for the shear strain instead of γ in the Newton law. The thermodynamic consistency can be checked for the Newtonian viscous element as well:

$$\dot{U}_d = \langle \dot{\varepsilon}, \sigma \rangle_{1D} - 0 = \dot{\varepsilon} \cdot \sigma = \sigma^2 \geq 0. \tag{2.12}$$

Due to the continuity of $\varepsilon(t)$, the dissipation energy course can be evaluated by using Newton law (2.11) and in the form of the Stieltjes integral it is:

$$U_d(t) = \int_0^t \dot{\varepsilon}(\theta)\sigma(\theta)\mathrm{d}\theta = \int_0^{\varepsilon(t)} \eta\dot{\varepsilon}(\theta)\mathrm{d}\varepsilon(\theta) = \eta\Big(\varepsilon(t) - \varepsilon(0)\Big). \tag{2.13}$$

Viscous elements act as dampers within the rheological models. All energy received during loading is dissipated and no potential energy is stored, $U_p = 0$, in the case of an incompressible liquid. The deformation process is completely irreversible, see Figure 2.7, right.

Rigid plastic element (StV)

A rigid plastic element is usually graphically represented by

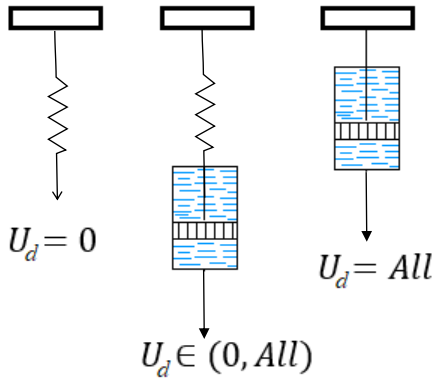

Figure 2.7: Uni-axial interpretation of the dissipated energy amount U_d in elastic element (left), in viscoelastic structure (middle), in viscous matter (right).

two plates superimposed on each other. The friction of a certain magnitude acts between both plates. The constitutive relation herein is not straightforwardly expressed. Saint-Venant plastic element (StV) exposed to a tensile or compressive load does not move at the beginning, until the moment in which the value of the stress reaches the threshold (tensile or compressive one). If so, the friction resistance is suddenly overcome and the material becomes plastic immediately, and its deformation increases linearly and unlimitedly up to the moment when the load decreases again under the threshold (tensile or compressive one). After that it remains unmoved again. This situation can be described mathematically by using the following variational inequality.

Definition 3
Let Z be the space of all admissible stress values with all thresholds situated on its boundary ∂Z. The plasticity is governed by the following physical principles:

- $\boldsymbol{\sigma} \in int(Z)$ ensures the persisting rigidity of the body
- $\boldsymbol{\sigma} \in \partial Z$ - the plastic behaviour is triggered

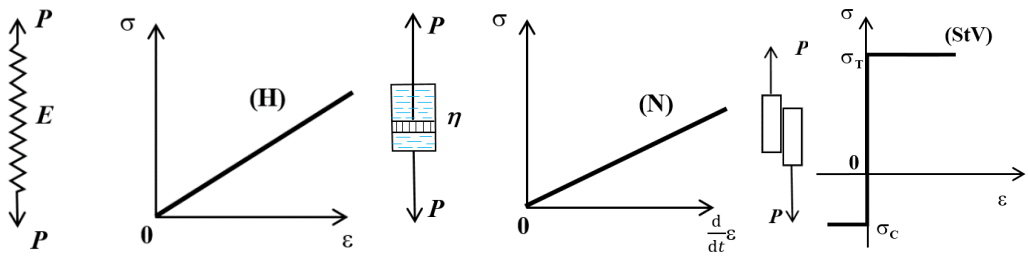

Figure 2.8: Pictograms and stress-strain graphs in elastic element (*H*) (left), viscous element (*N*) (middle) and rigid-plastic element (*StV*) (right), [19].

- $\langle \dot{\boldsymbol{\varepsilon}}, \boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}} \rangle \geq 0, \quad \forall \tilde{\boldsymbol{\sigma}} \in Z$

The last principle in Definition 3, is called the *Saint-Venant variational inequality* or *maximal dissipation rate principle* above the admissible stress values span. It states that while the threshold is not reached, the deformation does not change, i.e. $\forall \sigma \in int(Z) \Rightarrow \dot{\varepsilon} = 0$, [8, 20].
In the uni-axial case $Z = \langle -\sigma_C, \sigma_T \rangle$ and $\partial Z = \{-\sigma_C, \sigma_T\}$. We assume that σ_C, σ_T are two positive constants herein. Therefore, logically $0 \in Z$ and such a case corresponds to the natural hypothesis that no deformation occurs for $\sigma = 0$. This condition is essential for the thermodynamic consistency of the model. In Figure 2.8, the fifth and the sixth pictures, an uni-axial representation of a rigid-plastic body, the graphical interpretation of the stress-strain relation is performed. The polyline graph is called the *diagram of three branches*. Herein, as Z is an interval in one-dimension, its boundary - the set of its two endpoints - the negative compressive threshold $-\sigma_C$ and the tension threshold σ_T. In general $\sigma_C \neq \sigma_T$. When any of the thresholds is reached, the plasticity process starts and persists until the load magnitude comes back to the interval $(-\sigma_C, \sigma_T)$ again and the rigidity comes back, while the permanent (plastic) deformation remains. Since no potential energy is stored, i.e. $U_p = 0$, no recovery occurs. It has been observed in the laboratory tests that during plastic deformation the volume change is negligible [37].

In Chapter 2.1.2 both elastic and viscous elements are introduced together with their physical properties. Let us additionally emphasize that during the loading/unloading process:
An **ideal elastic material** performs no dissipation of the received mechanical energy, i.e. all energy gathered within the loading process is utilized to the process of the complete reversion towards the original stage when the load ceased and there is no phase shift between the deformation and the stress.
An **incompressible viscous material** is a liquid that flows fluently, with a constant velocity under constant load. The deformation is not reversible at all and the density of liquid remains the same during the time the load persists. Herein, the amount of dissipated energy quantifies the inner structural damping.
We can briefly summarize:

- elastic material ⇒ no dissipated energy
- viscoelastic materials ⇒ partial dissipation of energy
- viscous material ⇒ all energy dissipated

2.1.3 Two types of connection, structural forms and graphical interpretation of rheological models

There are two types of fundamental elements connection in rheological models: *in parallel* and *in series*. The connection is realized in a way that maintains the unique degree of freedom. The following considerations are provided for the one-dimensional case:

Parallel connection is abbreviated as "|" in the structural rheological forms. Both (or more) elements supposed to be connected *in parallel* are put one by one to keep a unique degree of the freedom and interconnected mutually by using two rigid parallel plates. These plates do not rotate or deform, and do not flex, they just move in line with the degree of the freedom of the raised model, i.e. perpendicularly to their longitudinal axis. The rigid plates are represented by the upper and the

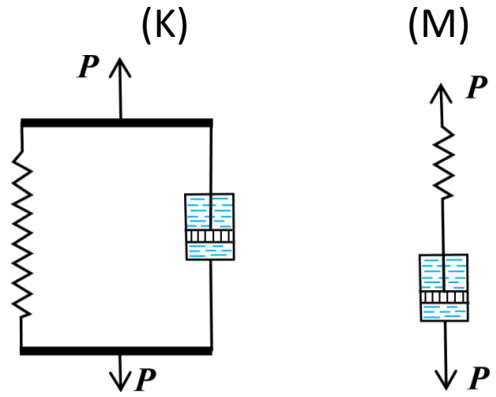

Figure 2.9: (K) - Kelvin-Voigt model (left) and (M) - Maxwell model (right). Parallel and serial connection of elementary members (H) and (N).

lower bold horizontal lines in Figure 2.9, left. Herein, the degree of the freedom is vertical. Logically, the movement of the parallel items, i.e. the displacement of both (or more) components coupled in parallel connection remains equal to each other during all loading and unloading processes. At the same time, the global stress which originates during loading and unloading is equal to the sum of the stress values in particular components. Since this consideration deals with geometry, corresponding equations are called geometric equations:

$$\begin{aligned} \varepsilon &= \varepsilon_H = \varepsilon_{StV}, \\ \sigma &= \sigma_H + \sigma_{StV}. \end{aligned} \tag{2.14}$$

Indices H, StV and N in (2.14) reflect the incidence of the particular variable with its element, e.g. ε_H is a deformation of the Hookean element within the configuration, etc. Apparently, (2.14) can be generalized to n members connected in series:

$$\begin{aligned} \varepsilon &= \varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_2 \\ \sigma &= \sigma_1 + \sigma_2 + \ldots + \sigma_n. \end{aligned} \tag{2.15}$$

Serial connection is abbreviated as "**-**" in the structural rheological forms. It is realized by joining the elements one after the

another by the direction of the degree of freedom, alongside the (supposed) acting load. The global deformation is then equal to the sum of deformations of the particular elements; while the stress is distributed to the particular elements evenly,

$$\begin{aligned} \varepsilon &= \varepsilon_H + \varepsilon_N, \\ \sigma &= \sigma_H = \sigma_N. \end{aligned} \tag{2.16}$$

An subsequently, for n members connected in series:

$$\begin{aligned} \varepsilon &= \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n, \\ \sigma &= \sigma_1 = \sigma_2 = \ldots = \sigma_n. \end{aligned} \tag{2.17}$$

The situation is depicted in Figure 2.10, right. It is evident that all geometric relations of each model, however complex, will be determined by the configuration of the model.
Having elementary matters and both types of connections at hand, we can create various configurations resulting in less or more complex rheological models, see chain and recurrent models in Chapter 3.2, 3.4 or 3.3. Taking abbreviations of elementary members and signs "|" and "-" standing for serial and parallel connection respectively, we can write down the structural form of any rheological model and by using pictograms we can draw a graphical scheme of it. In Figure 2.10 the model $(H)|(StV)$ is performed on the left and $(H)-(N)$ on the right-hand side.
If a rheological model includes only viscous and elastic elements, it is called viscoelastic. And, since viscoelastic materials are of great importance within the scope of material science, there exists a special subbranch of rheology exploring the viscoelastic models, called the viscoelasticity theory which deals with models involving both elastic and viscous matters in various configurations and no other rheological member is included.

2.2 Match between the material and its model

As has been documented before, alongside this book, constitutive relations of materials are a tool for further theoretical testing and predictions of their mechanical behaviour (response,

reaction) under various types of load. As it was already mentioned, viscoleastic models always simulate materials in a certain level of simplification. This is why the originating model has to take the future supposed load range of load into account. For example, when we want to test the operating regime of a specific type of concrete within its ordinary operating regime (safely between from it maturing and failure phase), we use a viscoelastic model. Otherwise, the plastic member has to be involved in this material rheological model.
The question of whether a model fits the material, can be hereafter answered in a the confrontation of the theoretical investigation results with the laboratory experimental results and the measurements. So, the feasibility of both the constitutive relation and the model itself can and has to be validated.
However, a much more essential problem appears very often, typical for various sciences and research - the *inverse task*. Often we have to find a well-fitting model for an explored material when we are equipped with a set of measured data.
Worldwide research groups have worked on that task for various types of rheological problems in recent decades, [10, 11, 12, 17, 20, 21, 50]. Moreover, due to both geometric and physical uncertainty, the problem can become very complex. The problem has got its name as well, as far as polymers are concerned, *Deborah number dilemma* - we are supposed to determine which theoretical models describing the viscoelastic flow of polymers correspond the best with the experimental findings. By now, dozens of models have been proposed but they always work only limitedly and not in general, [40]. However, if an experienced rheologist estimates the configuration, only the problem of the estimation of physical characteristics remains. Then an optimization algorithm, heuristic or meta-heuristic, Swarm algorithm, Gravitational search algorithm, Genetic algorithm, Artificial annealing or others are often used. e.g., [38, 36]. In Chapter 3.4.1, the Prony series approximation of the generalized Maxwell model is provided. The inverse task of obtaining the Prony coefficients is given by [33].

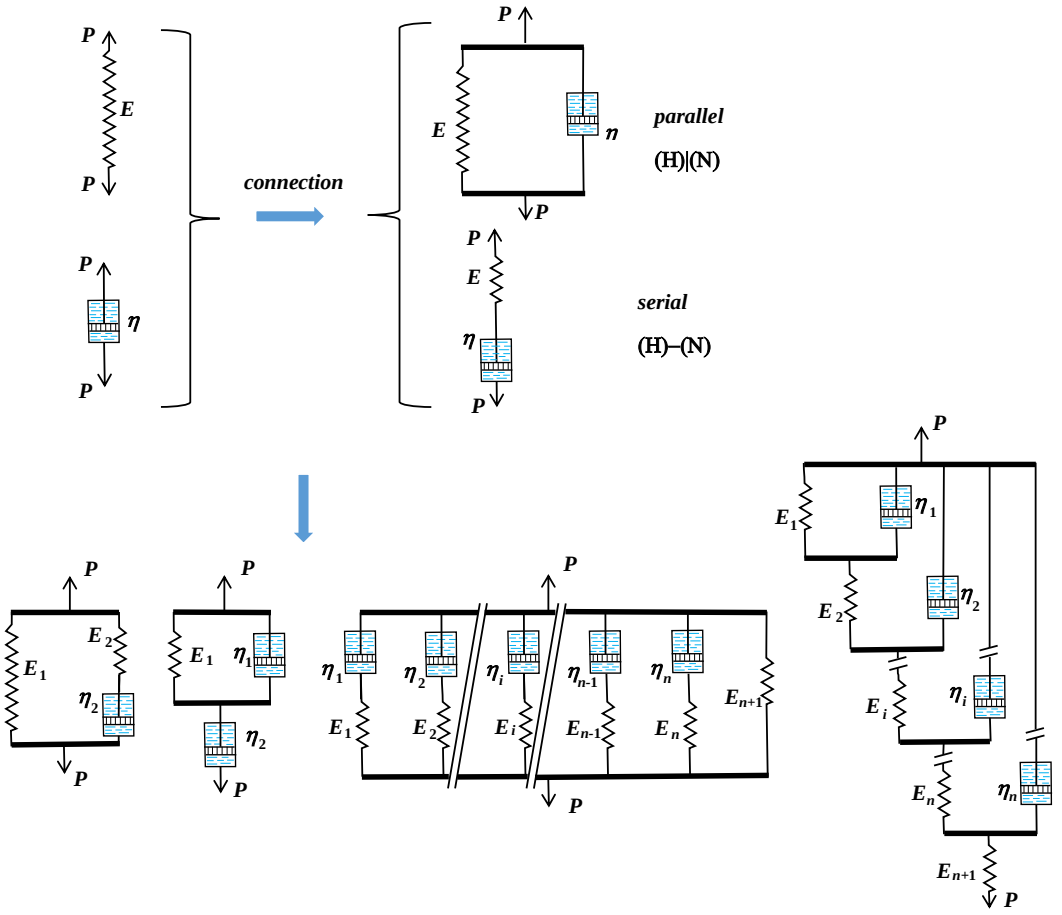

Figure 2.10: Viscoelastic model configuration. Parallel and serial connection. Simple (upper-right) and more complex (lower) rheological models.

Chapter 3

VISCOELASTICITY

3.1 Viscoelastic modelling

Each viscoelasticity property enfolds both viscosity and elasticity in a synergy. As it is apparent from its name, only the viscous and elastic fundamental elements are present in any viscoelastic model. Viscoelastic behaviour is exhibited by its damping ability and hysteresis involved.
Having two fundamental elements (*N*) and (*H*) at hand, we can exploit parallel and/or serial types of connection and to create various configurations and various viscoelastic models. The constitutive relation of the entire model is then determined by the (elementary) constitutive relations of (*H*) and (*N*); and by the geometry following the configuration. The basic geometric relations (2.14) and (2.16) for the connection *in series* and *in parallel* are always used.

3.1.1 Linear viscoelastic model

The response of a viscoelastic medium to a load represented by an appropriate viscoelastic model, can be generally expressed in the form [45]

$$F(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}}, \ddot{\boldsymbol{\varepsilon}}, \ddot{\boldsymbol{\sigma}}, \ldots, \boldsymbol{r}, \theta, p, t, \tau) = 0, \tag{3.1}$$

with $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ being the stress and strain tensors respectively, $\boldsymbol{r}$ - position vector, p - pressure, t - time, τ - action history

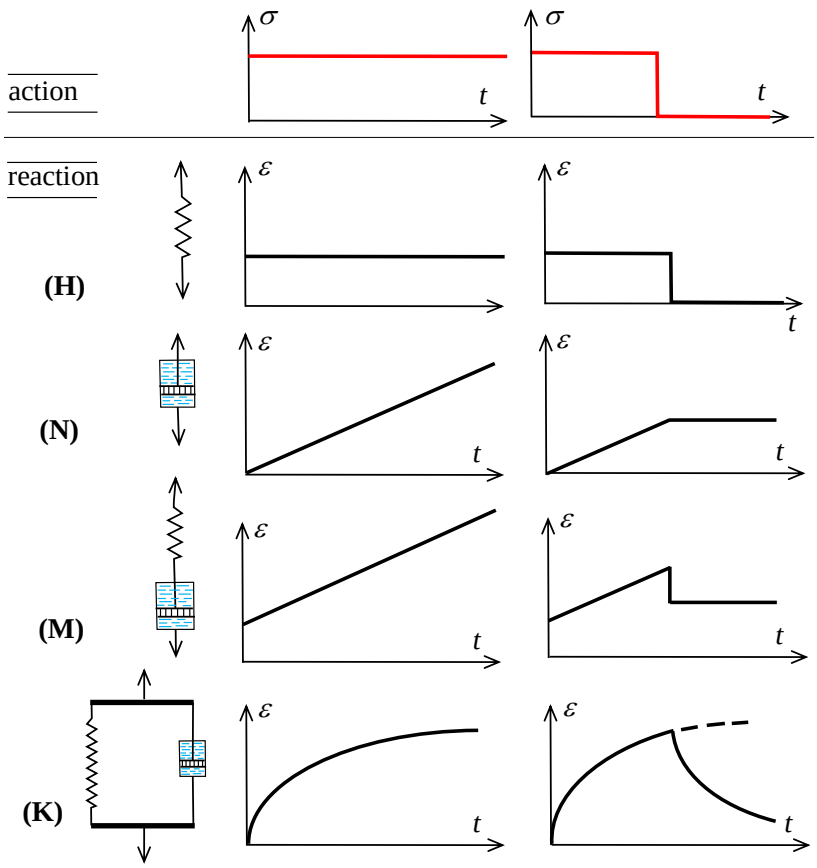

Figure 3.1: Examples of action and reaction functions on elementary matters (*H*), (*N*) and two element viscoelastic models (*M*), (*K*).

time, θ - temperature. Both $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\varepsilon}(t)$ are tensor functions of time and their time derivatives appear in the viscoelastic model. Since the impact of the mechanical load in viscoelastic materials can persist in time more or less intensively even after the load ceases or changes, it is important to record the entire load history (or the load history impact) and take it into account to specify the current mechanical response. The action history time τ records all significant changes in the action load and their impact on the reaction (response) function. Therefore, each time t_0 when we start with the exploration of the mechanical response (action - reaction), we have to take the current

situation $R(t_0)$ into account, i.e. $R(t_0)$ stands as the initial stage of the exploration.
In the one-dimensional case when the physical situation exhibits

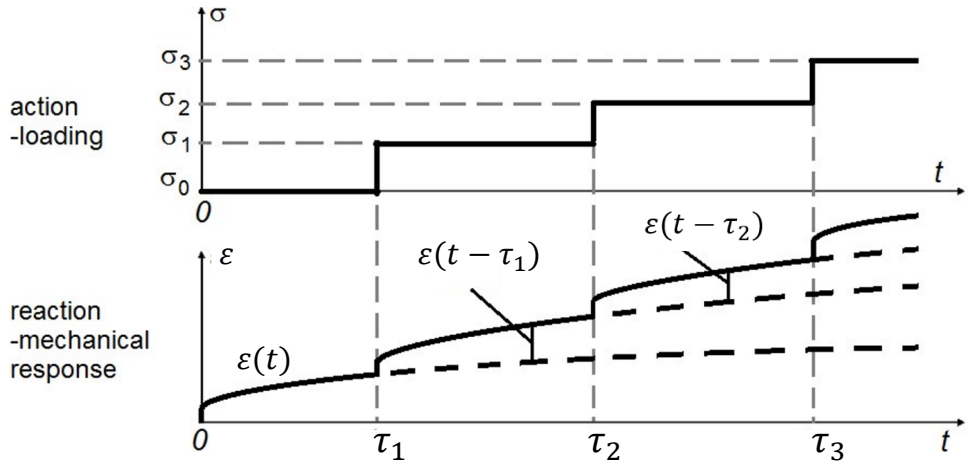

Figure 3.2: Superposition performance of the creep function. The action: a stepwise stress function (upper graph). The reaction: the particular creep functions added one after the another (superpositioned) to the resulting global strain function (lower graph)

negligible influences of the temperature and pressure, (3.1) reduces to

$$F(\sigma, \varepsilon, \dot{\sigma}, \dot{\varepsilon}, \ddot{\varepsilon}, \ddot{\sigma}, \ldots, t, \tau) = 0. \tag{3.2}$$

The highest order time derivative in the constitutive equation (3.2) corresponds to the number of the so-called "irreducible" viscous elements included in a viscoelastic model configuration. Indeed, the "reducible" viscous elements are those elements that can be formally replaced by one unique element within the configuration. It means that the mechanical response of the viscoelastic model does not change when the reducible viscous elements are reduced to one. Reducible elements are dealt in Chapter 3.4.1. Therein, in Figure 3.17, there are examples of reducible viscous elements and forms (3.111) are provided, specifying the new arisen global physical parameter η which couples the particular viscous elements after parallel and serial reducing, respectively.

Within the viscoelasticity theory, the reaction R depends on the action A and its history, too, [45].

$$R = R\big[A(t,\tau)\big], \quad \tau < t. \tag{3.3}$$

Herein, the functional R is supposed to be linear, which means that the process of loading is "sufficiently slow" so that no microcracks in the material appear. Besides, as has been already mentioned, the Boltzmann principle of superposition is valid in the linear case. If R is time translation invariant in addition, we can write

$$R = R\big[A(t-\tau)\big], \quad \tau < t. \tag{3.4}$$

Relation (3.4) refers to a material "with memory".
If the material is time translation variant, (3.3), the current mechanical response is influenced by the history of the load, i.e. type of load and duration. If it is not, the material represented by the viscoelastic model behaves in the same manner at any time. The same action load and the same initial value at any moment of reaction function yield the same resulting reaction.
As soon as the constitutive equation of a model is accomplished, the various action-reaction processes can be simulated, computed, and graphically interpreted; so the material behaviour under various types of load can be explored, and examined, and a prediction of the material's behaviour can be provided.

3.1.2 Creep and relaxation of the material. Creep test, relaxation test and time dependent material characteristics

In general, creep is a typical exhibition of a viscoelastic material as a response to a persisting constant tensile or compressive load imposed. It is a continuation of the deformation process despite the stress load remaining unchanged. Adequately, the relaxation starts immediately after a constant deformation is imposed on a viscoelastic material and maintained in time. The creep compliance $C(t)$ and the relaxation modulus $E(t)$ are so-called time-dependent material characteristics that can be stipulated by executing creep and relaxation tests.

Both $C(t)$ and $E(t)$ reflect some information about the age of the focused materials and facilitate eventual primary comparison and categorization of viscoelastic materials. *Creep and relaxation tests* (yielding corresponding constitutive equations, the special forms of (3.1) or (3.3) or (3.4)) are two typical tests used for mutual comparison of material in the sense of mechanical response to the load. As soon as the action function is given, it incomes to the *constitutive relation*, which becomes the *constitutive equation*, a linear differential equation solvable by standard mathematical methods. In Figure 3.2, the graphical interpretation can be seen. The stepwise constant stress action evokes the strain reaction herein. The strain in this case is a superposition of the particular stresses $\varepsilon(t-t_i)$ operating at the particular intervals $\langle t_{i-1}, t_i\rangle$.

Creep test

We impose an *immediate stress load* σ^* on the material represented by its viscoelastic model instantaneously and keep it unchanged by a lapse of time. Despite the constant stress, the strain continues to increase. Such a response of the viscoelastic material is called the *creep* of the material. Mathematically - when substituting the function of stress representing the action to (3.3) and (3.4), the strain (function in time) is the reaction.

Relaxation test

We impose an *immediate strain* (action ε^*) on the viscoelastic medium represented by the viscoelastic model and maintain the load unchanged over time. The responding stress magnitude decreases as time lapses, and the material is *relaxing* despite the persisting constant load. Mathematically - as soon as the strain function is used as the action in (3.3) and (3.4), the stress function is the reaction.
The creep and relaxation tests carried out for specific viscoelastic models are provided in Chapter 3.2.4.
In the case of a stepwise action function, e.g. immediate load imposing and ceasing, it is worth employing the Heaviside func-

tion

$$h(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases} \tag{3.5}$$

All resulting reactions of the preceding time subinterval action input to the consequential process as initial values. Keeping the superposition principle still in mind, we can express the resulting reaction as the sum of all previous particular actions' contributions [45]:

$$R(t) = \Delta A(\tau_1) R_h(t - \tau_1) + \Delta A(\tau_2) R_h(t - \tau_2) + \ldots$$

$$\cdots + \Delta A(\tau_k) R_h(t - \tau_k), \tag{3.6}$$

where $\Delta A(\tau_i) = A(t)$ for $t \in \langle \tau_{i-1}, \tau_i)$ and the shortening $R_h(t - \tau_i) = R[A(t)h(t - \tau_i)]$ with $\tau_i < t$ is used. When we shorten the length of each subinterval $\langle \tau_{i-1}, \tau_i \rangle$, even after taking the limit of its length going to zero, the action becomes a continuous function and the reaction (3.6) takes the integral form

$$R(t) = \int_0^t R_h(t - \tau) \frac{\mathrm{d}A(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau. \tag{3.7}$$

Time-dependent material characteristics

The Hook's law (2.7), i.e. the linear stress on the strain relation, can be expressed in the inverse form, the strain on the stress dependence

$$\varepsilon = C\sigma \tag{3.8}$$

with $C = \frac{1}{E}$ being the so-called *compliance modulus*, the reciprocal value to the elastic modulus. Apparently, C is a constant when E is constant. Within the viscoelasticity theory, outside of the linear stress-strain relation we will use a viscoelastic generalization of both (2.7) and (3.8) in the sense of the claim: *compliance modulus* = strain function/stress function, and *stiffness modulus* = stress function/strain function. If we take e.g. a creep test, i.e. action imposed is a constant stress σ^* kept in time, in which we have derived the resulting strain function

$\varepsilon(t)$, then the time-dependent creep *compliance modulus* of the material corresponding to this action is

$$C(t) = \frac{\varepsilon(t)}{\sigma^*}. \tag{3.9}$$

In the same manner, the *relaxation modulus* of the material is yielded when the relaxation test is carried out,

$$E(t) = \frac{\sigma(t)}{\varepsilon^*}, \tag{3.10}$$

with a constant deformation ε^* maintained during the entire relaxation test and with the resulting stress reaction $\sigma(t)$ computed from the constitutive equation.
Herein, $C(t)$ and $E(t)$ are time-dependent functions that stand as physical (time-dependent) material characteristics. Hence, the first and basic mutual comparison of materials being developed can be done.

3.1.3 Derivation techniques of constitutive relations for two-element viscoelastic models

Each viscoelastic model involves at least one elastic element (H) joined with at least one viscous element (N) (at least two different members in total). A more complex viscoelastic model is often regarded as a system of submodels connected *in parallel* or *in series*. As far as the constitutive equations of a complex viscoelastic model are concerned, such a split enables us to proceed conveniently step by step; first treating each submodel separately and doing an appropriate completion afterwards.
Viscous and elastic matter are elementary members of each viscoelastic model. Thus, all particular physical relations have to be taken into account together with all geometric relations while deriving the global constitutive relation of such a model. After doing so, the global stress-strain relation can be derived. In the following, both two-element models are treated in this manner.

Kelvin-Voigt model constitutive equation

In the Kelvin-Voigt model $(K) = (H)|(N)$, one viscous and one elastic element are connected *in parallel*. Two parallel rigid plates are used to align and restrict the movement of entire model in the direction of the unique degree of freedom. This is graphically represented by two bold horizontal lines in Figure 2.9, left, that move in the direction perpendicular to their longitudinal axes of both of them, i.e. up and down in this figure, aligned with the direction of the acting load. This ensures a common displacement of both (or more) coupled members. To establish the global constitutive relation of the Kelvin-Voigt model we are first supposed to collect the corresponding geometric formulas following from the parallel connection (2.14) and the physical relation of (H) and (N), i.e. (2.7) and (2.11). Let us recall that by the subindexes H or N, the incidence of the particular variable with (H) and (N) elements is assessed.

$$\varepsilon = \varepsilon_H = \varepsilon_N, \tag{3.11}$$

$$\sigma = \sigma_H + \sigma_N, \tag{3.12}$$

$$\sigma_H = E\varepsilon_H, \tag{3.13}$$

$$\sigma_N = \eta\dot{\varepsilon}_N. \tag{3.14}$$

We aim to extract the relation between the global stress function $\sigma(t)$ and the global strain function $\varepsilon(t)$ of the entire model, expressed by physical characteristics E and η only. It means to couple equations of (3.11) - (3.14) in one in a way that all indexed variables $\varepsilon_H, \varepsilon_N, \sigma_H$ and σ_N will be eliminated from this system. From the equation (3.11) we can see that the indexes of strain variables ε can be simply omitted. Then, in the equation (3.12), we can substitute for σ_H and σ_N from the equation (3.13) and (3.14). This handling yields the required global physical (constitutive) equation of the Kelvin-Voigt model:

$$\sigma = E\varepsilon + \eta\dot{\varepsilon} \tag{3.15}$$

Maxwell model constitutive equation

The Maxwell model $(M) = (H) - (N)$ is created by the serial connection of one viscous and one elastic element, Figure 2.9, right. In the system of geometric and physical equations attached to this model

$$\varepsilon = \varepsilon_H + \varepsilon_N, \tag{3.16}$$

$$\sigma = \sigma_H = \sigma_N, \tag{3.17}$$

$$\sigma_H = E\varepsilon_H, \tag{3.18}$$

$$\sigma_N = \eta\dot{\varepsilon}_N, \tag{3.19}$$

it can be seen that the strain values are summed up and the stress is distributed evenly between both members. We are again supposed to eliminate $\varepsilon_H, \varepsilon_N, \sigma_H, \sigma_N$ from the system (3.16) - (3.19) to obtain the relation between the global stress function and the global strain function expressed using physical characteristics only. Due to the second equation, we can omit the indexes in the stress variables. From (3.18), we can express $\boldsymbol{\varepsilon}_H = \frac{\sigma_H}{E}$ and from (3.16), we have $\dot{\boldsymbol{\varepsilon}}_N = \frac{\sigma_N}{\eta}$. Since we do not have an explicit form of $\boldsymbol{\varepsilon}_N$, we will differentiate the equation (3.16) before substituting to it. So we substitute both for $\dot{\varepsilon}_H = \frac{\dot{\sigma}_H}{E}$ and $\dot{\varepsilon}_N = \frac{\varepsilon_N}{\eta}$. The resulting global physical relation of the Maxwell model is then

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{3.20}$$

Both two-element models, (K) and (M) are regarded as very basic viscoelastic models, and as already mentioned above, together with (H) and (N), very often enter to more complex rheological models as submodels. Accordingly, the Zener model $(Z) = (H_1)|[(H_2) - (N)]$ can be written in a shortened form as $(Z) = (H_1)|(M)$, etc. (see the three-dimensional model in Chapter 3.2).

As has been already mentioned above, during the loading and unloading process of a rheological material, we can observe a discrepancy in the mechanical response: the same action value evokes different reaction value during the loading process in comparison to the unloading process. This phenomenon is called the *hysteresis* [21]. It is usually caused by the dissipation of energy (see Figure 2.3).

3.2 Conditional stiffness

In order to further automate the derivation of constitutive relations, we should try to generalize and facilitate the process of deriving the global constitutive equation of a model of a certain configuration.
As we have already detected in the previous text of this book, due to the presence of the Newton's viscous element in each viscoelastic model, the time derivative is always involved in constitutive relations of these models.
For the sake of future better handling, we employ the (linear) differential operator $D = \frac{\mathrm{d}}{\mathrm{d}t}$ in viscous element stress-strain equation (2.11) first. Then the physical equation of (N) acquires the form

$$\sigma_N = \eta D \varepsilon_N \tag{3.21}$$

and couples the physical parameter η with the differential operator $E_{visc} = E_{visc}(D) = \eta D$ getting

$$\sigma_N = E_{visc} \varepsilon_N. \tag{3.22}$$

The operator $E_{visc} : C_1(R) \to C_0(R)$ designated in (3.22) is called the *conditional stiffness* of the uni-axial viscous matter (N). Herein $C_0(R)$ stands as the space of all continuous functions over real variables and $C_1(R)$ the space of all continuous functions over real variables whose derivatives are continuous as well.
Having the differential operator within E_{visc} in mind, from now on we can regard the constitutive relation (3.21) as a generalized Hook's law. For the sake of completion, we naturally put $E_{elast} = E$ as the conditional stiffness of Hookean matter (H) in an uni-axial case as well. Having $E_{elast} = E$ and $E_{visc} = \eta D$ at hand, we can write down the constitutive relation (3.15) and (3.20) of both two-element viscoelastic models (K) and (M) in the sense of conditional stiffness. The transcription of the constitutive relation of the Kelvin-Voigt model (3.15) is then

$$\sigma = (E + \eta D)\varepsilon. \tag{3.23}$$

Let us moreover designate the operator

$$\hat{E} = \hat{E}(D) = (E + \eta D) \tag{3.24}$$

as the global *conditional stiffness* of the model (*K*). In such a case, the constitutive equation of (*K*) takes the form

$$\sigma = \hat{E}\varepsilon, \tag{3.25}$$

that stands as a generalization of the Hook's law as well.
Similarly, the constitutive relation (3.20) of (*M*) can be rewritten in the sense of conditional stiffness as

$$D\varepsilon = \left(\frac{1}{E}D + \frac{1}{\eta}\right)\sigma \quad \rightarrow \quad \varepsilon = \left(\frac{1}{E} + \frac{1}{\eta D}\right)\sigma = \frac{1}{\hat{E}}\sigma \tag{3.26}$$

from where we can see the constitutive relation of (*M*) is again of the form (3.25). Moreover, from (3.26) we can withdraw the formula for the global conditional stiffness coefficient of (*M*) coupling both the viscous and the elastic conditional stiffness parameters into one, $\hat{E}$, given by the form:

$$\frac{1}{\hat{E}} = \frac{1}{E} + \frac{1}{\eta D}. \tag{3.27}$$

Thereafter, (3.27) represents a general rule for the coupling of arbitrary submodels (two or more) connected *in series*.
By utilising conditional stiffness conception, the global constitutive equation of any uni-axial viscoelastic model, however complex, can be expressed in the form of a generalized Hook's law (3.25).
Let us recall that $\hat{E} = \hat{E}(D)$ includes all geometric and atomic physical relations of the model:

- In (3.23), we have the global conditional stiffness coefficient of (*K*). It is the plain sum of the conditional stiffness parameters of the particular elements (*H*) and (*N*).

- In (3.27), the conditional stiffness of (*M*) is derived, since we have the reciprocal value of the global conditional stiffness equal to the sum of the reciprocal conditional stiffness values of particular members (*H*) and (*N*).

Naturally and easily the enhance of the conditional stiffness computation rules for *n* members can be done. In the case

of a parallel connection of n members, the resulting conditional stiffness is equal to the plain sum of conditional stiffness parameters of particular members,

$$\hat{E} = \sum_{i=1}^{n} E'_i \tag{3.28}$$

and, in the case of a serial connection, the total conditional stiffness $\hat{E}$ should be expressed from the equation

$$\frac{1}{\hat{E}} = \sum_{i=1}^{n} \frac{1}{E'_i} \tag{3.29}$$

where E'_i is the particular conditional stiffness of i^{th} particular member. At first glance, regarding parallel and serial connection, it is natural to anticipate an analogy between the handling of viscoelastic models and the handling of electrical circuits.
Let us emphasize that $\hat{E} = \hat{E}(D)$ stands as a general form of conditional stiffness and in one dimension $\hat{E}$ is a scalar differential operator while in more dimensions it is a tensor differential operator.

Three-element viscoelastic models

Three elements models are widely used in simulating materials within the scope of the viscoleasticity theory. In Figure 3.3 there are some examples of three-element models depicted which are of frequent use in practical rheology of materials: The Poynting - Thompson model (PTh) which is often used as the model of concrete under an ordinary load, Zener model (Z), Schofield (Sch) and ("4"), without a name up to now. All those three element models are widely used in the biomechanical or tissue engineering investigation. By using the elimination techniques elaborated in Chapter 3.1.3 or by using the conditional stiffness tool together with the rules (3.28) and (3.29) for the parallel and serial connection respectively, we can accomplish the constitutive relations representing each model.
Next, the structural form (sf), both in the sense of atomic members or in the sense of submodels and constitutive relation in implicit form (cr) are provided:

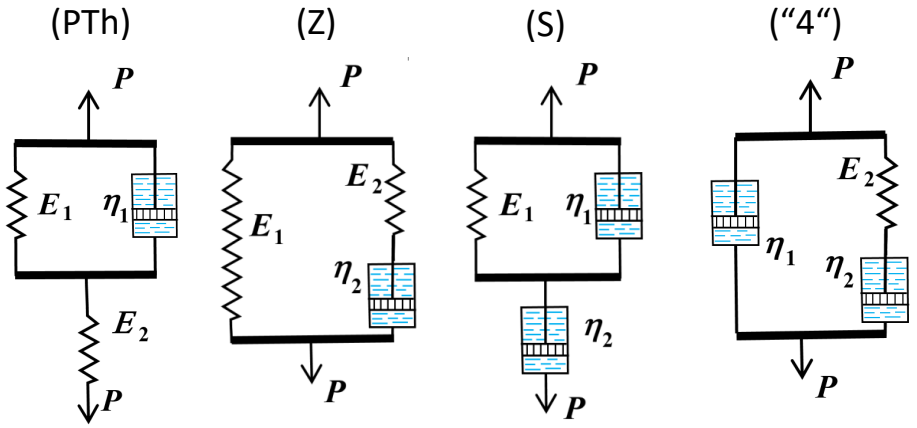

Figure 3.3: Three element viscoelastic models: (PTh) - Poynting-Thompson model, (Z) - Zener model, (S) - Schofield model and ("4") [29].

- sf: $(PTh) = (H_2) - [(H_1)|(N)] = (H_2) - (K)$
  cr: $E_1\eta_1\dot{\varepsilon} + E_1E_2\varepsilon = \eta_1\dot{\sigma} + (E_1 + E_2)\sigma$
- sf: $(Z) = (H_1)|[(H_2) - (N)] = (H_1)|(M)$
  cr: $\eta_1(E_1 + E_2)\dot{\varepsilon} + E_1E_2\varepsilon = \eta_1\dot{\sigma} + E_2\sigma$
- sf: $(S) = (N_2) - [(H)|(N_1)] = (N_2) - (K)$
  cr: $\eta_1\eta_2\ddot{\varepsilon} + E_1\eta_2\dot{\varepsilon} = E_1\sigma + (\eta_1 + \eta_2)\dot{\sigma}$
- sf: $("4") = (N_1)|[(H) - (N_2)] = (N_1)|(M)$
  cr: $\eta_1\eta_2\ddot{\varepsilon} + E_1(\eta_1 + \eta_2)\dot{\varepsilon} = E_1\sigma + \eta_2\dot{\sigma}$

Recalling the fact that the number of irreducible dash-pots involved in a viscoelastic model determines the order of its constitutive differential equation, the reader can review in Figure 3.3 that the order will be 1 in the cases (PTh) and (Z); and 2 in (S) and ("4").

3.2.1 Ordinary differential equations with constant coefficients

Along this book, the ordinary differential equations with constant real coefficients stand as the constitutive equations of vis-

coelastic models. Moreover, the roots of the attached characteristic equations are focused since they are directly connected with practice; they are related to the relaxation and retardation time values that can be measured in lab. That is why this small chapter is provided herein, recalling the linear ordinary differential equations, general solutions, characteristic equations and their roots.
Keeping in mind the further generalization for n^{th} order, let us start with the linear ordinary differential equation of the second order

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = f(x), \tag{3.30}$$

where $a_0(x), a_1(x)$ are continuous functions over the interval, over which the solution is sought. The function $f(x)$ is called right hand side.
The solution $y(x)$ of (3.30) is always of the form

$$y(x) = y_h(x) + y_p, \tag{3.31}$$

where $y_h(x)$ is the *homogeneous solution*, i.e. general solution of the attached homogeneous equation

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0 \tag{3.32}$$

and a particular solution $y_p(x)$ of (3.30) which can be found by the method of variation of the constants. Let us recall that each linear homogeneous ordinary differential equation has the trivial solution $y(x) = 0$.
If we take constants a_0, a_1 instead of the functions $a_0(x), a_1(x) \in \mathcal{R}$ in (3.30), we get the linear differential equation with constant coefficients

$$y''(x) + a_1y'(x) + a_0y(x) = 0, \tag{3.33}$$

whose non trivial solution is expected in the form of a linear combination of exponential functions. Let us take one:
$y(x) = e^{rx} \Rightarrow y'(x) = re^{rx} \Rightarrow y''(x) = r^2e^{rx}$ and supply it in (3.33). This yields $e^{rx}(r^2 + a_1r + a_0) = 0$. and apparently we can truncate e^{rx} concluding that in order to solve the differential equations with constant coefficients (3.33), we are supposed first to solve the characteristic equation

$$r^2 + a_1r + a_0 = 0. \tag{3.34}$$

One can easily see that this attempt is general for the n^{th} order of the differential equation, where the arisen characteristic equation is of the n^{th} degree.
Depending on the coefficients, solving (3.34) yields r_1, r_2 which are either two different real roots, or a pair of complex conjugate, or one double root r of the characteristic equation (3.34). Depending on this, we have

- $y_1 = e^{r_1 x}, y_2 = e^{r_2 x}$, if $r_1, r_2 \in \mathcal{R}$ are distinct,
- $y_1 = e^{\alpha x}\cos\beta x, y_2 = e^{\alpha r}\sin\beta x$, if $r_{12} = \alpha \pm \mathrm{i}\beta$,
- $y_1 = e^{rx}, y_2 = xe^{rx}$, if $r_1 = r_2 = r$, $r \in \mathcal{R}$.

Then

$$y_h(x) = c_1 y_1 + c_2 y_2. \tag{3.35}$$

Next, the general solution $y(x)$ of (3.30) can be computed by variation of parameters c_1 and c_2. Indeed, we replace c_1 and c_2. in by $v_1(x)y_1 + v_2(x)y_2$, and seek the solution to (3.30) in the form

$$y = v_1 y_1 + v_2 y_2, \tag{3.36}$$

where v_1, v_2 are designated functions to be determined. Two unknown functions v_1, v_2 should satisfy one equation

$$y''(x) + a_1 y'(x) + a_0 y(x) = f(x), \tag{3.37}$$

thus we can establish one condition more in order to make the solution of (3.37) more easy. Such condition is

$$v_1' y_1 + v_2' y_2 = 0$$

since it shortens the expression of y'. Subsequently we are supposed to solve the linear system of equations

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = f(x).$$

Finally, the functions v_1, v_2 can be expressed as:

$$v_1 = \int \frac{W_1}{W} \mathrm{d}x,$$

$$v_2 = \int \frac{W_2}{W} \mathrm{d}x,$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \; W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \; W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}.$$

More details about the solution of linear ordinary differential equations can be found e.g. in [14].

3.2.2 Constitutive equations, general solution, creep compliance modulus and relaxation modulus calculation

In the previous chapters, we have elaborated the tools for the *constitutive relations* computation for an arbitrary uni-axial viscoelastic model.
Given the constitutive relation, the *constitutive equation* arises by subjecting the model to any specific load, the action function. In this manner, the known *action* function arises (stress or strain, respectively) in the specified period and the *constitutive relation turns to the constitutive equation*, a differential equation with one unknown function. Then the solution to the constitutive equation is the reaction (strain or stress respectively) within the same time range. Of course, for this ordinary differential equation of n^{th}-order, n additional conditions are needed in order to get the unique solution. Usually, we have initial conditions at our disposal.

Example
Let us take the Maxwell model (**M**) with its constitutive relation (3.20) and let us

a) subject it to a *creep test with cease*. Instantly, starting in the time instant $t_0 = 0$, we burden the constant stress action σ^* and maintain it for $t \in \langle t_0, t_1 \rangle$. Next, at time t_1 let cease the load and keep the unloading period lapsing,

$\sigma(t) = 0$ for $t > t_1$.
The *creep compliance modulus* $C(t)$ derivation, $t \in \langle t_0, t_1 \rangle$, starts with substituting $\sigma(t) = \sigma^*$ into the implicit constitutive relation of (M) (3.20). Hereby we obtain the general solution to this equation simply by its plain integrating:

$$\dot{\varepsilon}(t) = \frac{\sigma^*}{\eta} \quad \Rightarrow \quad \varepsilon(t) = \frac{\sigma^*}{\eta} t + c \tag{3.38}$$

with c being an integration constant. Regarding the initial deformation $\varepsilon(t_0) = \varepsilon(0) = \varepsilon_{elastic}(t_0) = \frac{\sigma^*}{E_0}$, we compute c, thus the exact solution - creep function of (M) subjected to the creep test will be of the form

$$\varepsilon(t) = \frac{\sigma^*}{\eta} t + \frac{\sigma^*}{E_0}. \tag{3.39}$$

In this example, in order to emphasise the difference between the Young modulus of elasticity and the relaxation modulus derived in b), we use E_0 instead of E, ordinarily used along this book.
We can physically summarise: Initially at the time instant $t_0 = 0$ when the immediate stress σ^* starts to act on (M), an elastic deformation occurs instantaneously. But, exactly from this moment, a viscous deformation starts to increase in time without a limitation as long as the action is maintained. Finally, the creep compliance modulus of (M) subjected to the creep test is evaluated, see (3.9):

$$C(t) = \frac{\varepsilon(t)}{\sigma^*} = \frac{t}{\eta} + \frac{1}{E_0}. \tag{3.40}$$

After the stress load (immediately at t_1) removal the elastic deformation recovers (immediately at t_1), but a permanent deformation remains, due to the viscous element present in the model, connected *in series* (see Figure 3.1, the third model).
Mathematically, after the immediate elastic stress drop down to zero at the instant t_1, $\varepsilon(t_{1+})$ becomes the initial condition of the unloading branch, while the governing equation remains the same as in the loading period, i.e. (3.38),

which yields the constant leg of strain function from the instant t_1 onwards.

b) subject it to a *relaxation test* by immediate exposure (M) to a constant strain $\varepsilon(t) = \varepsilon^*, t \geq t_0$ and compute the *relaxation modulus* $E(t)$:
When substituting $\varepsilon(t) = \varepsilon^*$ into the implicit constitutive relation of (M), (3.20), we get a linear homogeneous differential equation

$$0 = \frac{\dot{\sigma}}{E_0} + \frac{\sigma}{\eta}. \tag{3.41}$$

The general solution is

$$\sigma(t) = ce^{-\frac{E_0}{\eta}t}, \tag{3.42}$$

with c being an arbitrary constant.
If there is no pre-stress supposed, we can observe that at $t_0 = 0$:

$$\varepsilon^* = \varepsilon_H(0) = \frac{\sigma_0}{E_0}.$$

This happens whenever a single spring is joined *in series* with the rest of the model. The initial stress is $\sigma(0) = \varepsilon^* E_0$. Exploiting this initial condition in (3.42) yields the precise solution, the relaxation function of (M) subjected to a relaxation test

$$\sigma(t) = \varepsilon^* E_0 e^{-\frac{E}{\eta}t}. \tag{3.43}$$

Finally, by using (3.10), the relaxation modulus of (M) subjected to the relaxation test, can be computed:

$$E(t) = \frac{\sigma(t)}{\varepsilon^*} = E_0 e^{-\frac{E_0}{\eta}t}. \tag{3.44}$$

It is worth to emphasise herein that both creep and relaxation moduli (time depending functions) arisen within the creep or relaxation tests respectively, are of the same type as the particular responding functions.

Initial conditions attached to a constitutive equation. Action and reaction history.

Based on the assumptions mentioned above, the resulting constitutive relation is of the linear differential form.
The linear ordinary differential equations with constant coefficients are utilized. The following paragraph (example) brings a short recall of these equations.
Given *an action, the constitutive relation* becomes the *constitutive equation*. The order of the differential equation is equal to the number of irreducible viscous elements involved in the model, (see the reducible elements in Figure 3.17 and the rules for reducing in Chapter 3.4). The general solution of a linear ordinary differential equation with constant coefficient and with a right-hand side can be computed by using e.g. the variation of constants method or the undetermined coefficient method in some special cases. Together with appropriate initial conditions, the exact solution, i.e. the sought *reaction* function can be computed, as well. Those *initial conditions* are of great interest in viscoelastic models. Namely in mechanical loading, not only the quantity of the load is decisive, but the duration of its acting as well. A residual reaction often persists even after ceasing the action load. That is why we have to take the action-reaction history carefully into account.
For the sake of better apprehension, let us utilize the considerations from the example above. By exposing the Maxwell model to a constant tensile force in a time interval, due to an elastic element connected in series in the model, an immediate deformation $\varepsilon_H(t_0)$ in the form of a jump occurs, being followed by a damped viscous irreversible deformation $\varepsilon_N(t), t > t_0$ that continues increasing until the load persists. It is evident that the *magnitude of this (viscous) irreversible deformation depends on the longitude of the action period* $\langle t_0, t_1 \rangle$. After the instantaneous load removal, the elastic deformation reverses immediately and the viscous deformation persists. This value has to input as the initial value of the further load process. That is why the so-called hereditary integrals are employed, as dealt with in the next chapter. As the action varies a lot within the life span of the viscoelastic matter, either the action course is traced and

recorded alongside the time interval one by one, and matched with the corresponding reaction, or written in the form of hereditary integral [31].
Besides, it is worth recalling the characteristic equation roots have their nice physical interpretation in the stipulation of retardation and the relaxation time spectra that can be gained directly by measurements in laboratory; hence a mutual confrontation is possible in this issue. The retardation and the relaxation phenomena are dealt in Chapter 3.2.4.

3.2.3 Hereditary integrals

In the previous text of the book the constitutive relations were written in a pure differential form, by which the stress-strain relation is usually given implicitly. As far as the action is given, together with an appropriate number of initial conditions, the reaction could be computed. But in many considerations, e.g. in the case of chemists dealing with polymers or polymer-based composites, there is a need to know the explicit expression of either stress on strain or strain on stress dependence, even prior to the action A is given. For this sake, the hereditary approach was developed, where the explicit relation is stipulated for a general action function. We utilize the functional $R(A)$ introduced in Chapter 3.1.1. The process of transforming the implicit constitutive equation to the general explicit form is demonstrated in the following example, where the functional $\mathcal{J}(\varepsilon) = \sigma(\varepsilon)$ is being derived for the Maxwell model.

Example
The task is to rewrite the implicit (differential) form of the constitutive relation (3.20) of the Maxwell model into the explicit form, namely as the stress on strain dependence or vice versa. The procedure is the same in both cases. So let us focus on the stress on strain dependence. Using the tools and concepts described in Chapter 3.1.1 we can extend the focused time interval $\langle 0, t\rangle$ by shifting the left bound down to $-\infty$. In practice, it means that we know the whole load history of the model as it was recorded from an initial time instant or we know its global

impact to the responding function at the initial moment of the current exploration.
In this example, in order to distinguish between the Young modulus of elasticity and the relaxation modulus, we use E_0 for denoting the Young modulus instead of E, ordinarily used along the book.
By using the integrating factor method with the integrating factor $e^{\frac{E_0 t}{\eta}}$, the constitutive relation (3.20) is converted into the integral form. So we then proceed with (3.20) in the form $dot\varepsilon E_0 = \dot{\sigma} + \sigma\frac{E_0}{\eta}\sigma$ and multiply it with this integrating factor, obtaining after a small rearrangement

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\frac{E_0}{\eta}t}\sigma(t)\Big) = E_0 e^{\frac{E_0}{\eta}t}\dot{\varepsilon}$$

and by integrating over the interval $\langle -\infty, \theta\rangle$ we have

$$\Big[e^{\frac{E_0}{\eta}t}\sigma(t)\Big]_{-\infty}^{\theta} = \int_{-\infty}^{\theta} E_0 e^{\frac{E_0}{\eta}t}\dot{\varepsilon}\ \mathrm{d}t \qquad \Rightarrow$$

$$\sigma(\theta) = \mathcal{J}\Big(\varepsilon(\theta)\Big) = \int_{-\infty}^{\theta} E_0 e^{\frac{E_0}{\eta}(\theta - t)}\dot{\varepsilon}\ \mathrm{d}t. \tag{3.45}$$

As we can see, the task is accomplished. In the functional (3.45) we have an explicit, integro-differential form of the constitutive equation equivalent to (3.20). To harmonize the notation we just change the names of variables $t \to \tau$ and $\theta \to t$. In this manner, we get a linear functional for the Maxwell model

$$\sigma(t) = \mathcal{J}\Big(\varepsilon(t)\Big) = \int_{-\infty}^{t} E_0 e^{-\frac{E_0}{\eta}(t-\tau)}\dot{\varepsilon}\ \mathrm{d}\tau \tag{3.46}$$

In the end, (3.46) can be written generally, in the sense of its relaxation modulus:

$$\sigma(t) = \mathcal{J}\Big(\varepsilon(t)\Big) = \int_{-\infty}^{t} E(t-\tau)\dot{\varepsilon}\ \mathrm{d}\tau \tag{3.47}$$

Looking back, we see (3.47) yields (3.46) for $E(t) = E_0 e^{-\frac{E_0}{\eta}t}$, which is exactly the relaxation modulus (3.44) of the Maxwell

model resulting from the relaxation test realized on (M).
The Heaviside function $h(t)$ is worth employing in the action history function, as was done in Chapter 3.1.2. Let $t_0 = 0$ be the starting point of our observation. Let us have the initial stress measured at t_0 and the relaxation modulus $E(t)$ of the model given. Then

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)h(\tau)\frac{\mathrm{d}\varepsilon(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau + \int_{-\infty}^{t} E(t-\tau)\varepsilon(\tau)\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau.$$

In the first integral, we can use the fact that $h(t) = 0$ for $\tau < 0$, in the second integral we have the Dirac delta function standing as the time derivative of the Heaviside function

$$\sigma(t) = \int_{0}^{t} E(t-\tau)\frac{\mathrm{d}\varepsilon(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau + \int_{-\infty}^{t} E(t-\tau)\varepsilon(\tau)\delta(\tau)\mathrm{d}\tau.$$

The second integral can be next evaluated by using the properties (3.54)-(3.56) of the Dirac delta function

$$\sigma(t) = E(t)\varepsilon(0) + \int_{0}^{t} E(t-\tau)\frac{\mathrm{d}\varepsilon(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{3.48}$$

If we want to write down the "pure" explicit stress on strain dependent form, we additionally rearrange the integral in (3.48) by using integration per partes and put the notation $\tilde{E}(t) = \dot{E}(t)$

$$\sigma(t) = E(0)\varepsilon(t) + \int_{0}^{t} \tilde{E}(t-\tau)\varepsilon(t)\mathrm{d}\tau. \tag{3.49}$$

In such an arrangement the time derivative moves from the strain action function into the time-dependent physical characteristics.
By the same mathematical tools, we can derive the inverse relation, the strain on stress explicit relation, valid for an arbitrary uni-axial viscoelastic model

$$\varepsilon(t) = C(0)\sigma(t) + \int_{0}^{t} \tilde{C}(t-\tau)\sigma(t)\mathrm{d}\tau \tag{3.50}$$

with the creep compliance function $C(t)$ obtained within the employed creep test, whereby $\tilde{C}(t) = \dot{C}(t)$.

Explicit form of constitutive equations. Stress on strain and strain on stress dependence

From the implicit constitutive relation (3.1) we can explicitly express either the stress reaction function $\sigma(t)$ dependent on a given strain action $\varepsilon(t)$, or the strain reaction function dependent on a given stress function.
Let us take the strain function as an action first. We already know that in general the resulting stress function is influenced not only by the current acting strain but also by the strain history. This can be formally written as

$$\boldsymbol{\sigma}(t) = \mathcal{G}_{-\infty}^{\infty}\left[\boldsymbol{\varepsilon}(t), \boldsymbol{\varepsilon}(t-\tau)\right], \tag{3.51}$$

where, $\mathcal{G}_{-\infty}^{\infty}[\,]$ is the so-called *constitutive functional*. It operates for $\tau \in \langle 0, \infty\rangle$ and $t \in (-\infty, \infty)$. It is a tensor valued functional which transforms the strain history $\varepsilon_{ij}(t)$ to the corresponding stress history $\sigma_{ij}(t)$. Apparently, the parametric dependence on strain is involved, the instantaneous response $\varepsilon(t)$ to an instantaneous load $\sigma(t)$, [7].
Assuming the linearity of the functional $\mathcal{G}$ and continuity of the strain history, we can use the Riesz representation theorem. For the continuous component functions of the tensor function E_{ijkl} in $(0, \infty)$, being also of a bounded variation (its total variation is bounded) on the arbitrary finite interval $\langle a, b\rangle \subset (0, \infty)$, we can use the Stieltjes form. Written component-wisely, it is:

$$\sigma_{ij}(t) = \int_0^{\infty} \varepsilon_{kl}(t-\tau)\mathrm{d}E_{ijkl}(\tau). \tag{3.52}$$

Herein, the total variation of a function on an interval $\langle a, b\rangle$ is taken:

$$V_a^b(f) = \sup_P \sum_{j=1}^{nP} \left|f(x_j) - f(x_{j-1})\right|,$$

where the supreme is evaluated over all possible partitions $P = \{a = x_0, x_1, \ldots, x_{nP} = b\}, x_0 \le x_1 \le \ldots \le x_{nP}$, of the interval $\langle a, b\rangle$. The 4^{th}-order tensor $\boldsymbol{E} = E_{ijkl}, (i, j, k, l = 1, 2, 3)$ is called *relaxation modulus*. Since the deformation tensor is symmetric, E_{ijkl} is symmetric too [7, 45]. Moreover, each component of

$E_{ijkl}(t) = 0$ for $-\infty < t < 0$.
However, since $\varepsilon_{kl}(t) = 0$ for $t < \tau$, the Heaviside function $h(t)$ can be used and in that case we can rewrite (3.52) and rearrange it as follows:

$$\sigma_{ij}(t) = \int_0^\infty h(t-\tau)\varepsilon_{kl}(t-\tau)\frac{\mathrm{d}(h(\tau)E_{ijkl}(\tau))}{\mathrm{d}\tau}\mathrm{d}\tau =$$

$$\int_0^\infty h(t-\tau)\varepsilon_{kl}(t-\tau)\delta(\tau)E_{ijkl}(\tau)\mathrm{d}\tau +$$

$$+\int_0^\infty h(t-\tau)\varepsilon_{kl}(t-\tau)\frac{\mathrm{d}E_{ijkl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{3.53}$$

Herein, we have exploited the relation of Heaviside function (3.5) with the Dirac function

$$\frac{\mathrm{d}h(\tau)}{\mathrm{d}\tau} = \delta(\tau), \tag{3.54}$$

and rearwards

$$\int_{-\infty}^\infty \delta(\tau)\mathrm{d}\tau = 1, \tag{3.55}$$

where

$$\delta(\tau) = \begin{cases} \infty, & \text{if } \tau = 0, \\ 0, & \text{if } \tau \neq 0 \end{cases}$$

is a generalized function called the Dirac function or *unit impulse*. As a corollary, we have

$$\int_{-\infty}^\infty f(\tau)\delta(\tau)\mathrm{d}\tau = f(0). \tag{3.56}$$

Recalling furthermore that $h(t-\tau) = 0$ for $\tau < t$ and using notation $f(0^+) = \lim_{x\to 0^+} f(x)$ the limit , we get

$$\sigma_{ij}(t) = \varepsilon_{kl}(t)E_{ijkl}(0^+) + \int_0^t \varepsilon_{kl}(t-\tau)\frac{\mathrm{d}E_{ijkl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau.$$

Next, we can proceed by integrating by parts in the second term and exploit the substitution $\tau = t - s$ afterwards:

$$\begin{aligned}
&\int_0^t \varepsilon_{kl}(t-\tau)\frac{\mathrm{d}E_{ijkl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau = \\
&= \left[\varepsilon_{kl}(t-\tau)E_{ijkl}(\tau)\right]_{0^+}^{t} - \int_0^t E_{ijkl}(\tau)\frac{\partial\varepsilon_{kl}(t-\tau)}{\partial\tau}\mathrm{d}\tau = \\
&= \varepsilon_{kl}(0^+)E_{ijkl}(\tau) - \varepsilon_{kl}(t)E_{ijkl}(0^+) - \int_0^t E_{ijkl}(\tau)\frac{\partial\varepsilon_{kl}(t-\tau)}{\partial\tau}\mathrm{d}\tau = \\
&= \varepsilon_{kl}(0^+)E_{ijkl}(\tau) - \varepsilon_{kl}(t)E_{ijkl}(0^+) + \int_0^t E_{ijkl}(t-s)\frac{\partial\varepsilon_{kl}(s)}{\partial s}\mathrm{d}s
\end{aligned} \tag{3.57}$$

Finally we get the **the explicit dependence of the stress on the strain**, expressed componentwisely:

$$\sigma_{ij}(t) = \varepsilon_{kl}(0^+)E_{ijkl}(\tau) + \int_0^t E_{ijkl}(t-\tau)\frac{\partial\varepsilon_{kl}(\tau)}{\partial\tau}\mathrm{d}\tau. \tag{3.58}$$

Relation (3.58) is valid for the tensor function $\varepsilon_{ijkl}(t)$, smooth in each component.
By the same consideration as described above, by the same procedure as for the stress on the strain relation, the inverse relation, the **strain on the stress explicit dependence** can be derived resulting in

$$\varepsilon_{ij}(t) = \int_{-\infty}^t C_{ijkl}(t-\tau)\frac{\partial\sigma_{kl}(\tau)}{\partial\tau}\mathrm{d}\tau \tag{3.59}$$

with $C_{ijkl}(t)$ being a creep compliance tensor coefficient. Material tensor functions $E_{ijkl}(t)$ and $C_{ijkl}(t)$ represent the mechanical response of the material represented by its viscoelastic model to a unit action - strain or stress, respectively.

3.2.4 Retardation and relaxation time spectra

The retardation and relaxation time spectra are regarded as additional material characteristics within the frame of viscoelas-

ticity.

Retardation time

Neither an elastic element nor any submodel connected to a dash-pot *in parallel* cannot carry out its immediate reaction because it is damped, i.e. *retarded* by the dash-pot. Within the creep test the difference between its virtual elastic deformation (without a dash-pot damper) and the real creep course is called *retardation*. Retardation changes over time. In the Kelvin-Voigt model (and this is so whenever (K) is involved as a submodel), the retardation expresses how the dash-pot restrains the spring in its potential elastic reaction. In the Poynting - Thompson model $(PTh) = ((H)|(N)) - (H)$ the spring connected *in series* works freely, while the spring connected *in parallel* with a dash-pot, is damped by this dash-pot, so the retardation increases gradually, (recall Figure 3.4, right).
The period in which the creep would reach the elastic value $\frac{\sigma^*}{E}$, when proceeding with its initial velocity is next of interest. It is the so-called *retardation time* of the material.
Geometrically performed, in the Kelvin-Voigt model we draw a tangent line to the creep curve at the initial point of the load. The intersection of this tangent line with the horizontal asymptote $\varepsilon_{elast}(t) = \frac{\sigma^*}{E}$ (supreme of the creep function), determines the point where the retardation time is measured. Its t abscissa yields the endpoint of the retardation time. Reader is encouraged to check the analogous situation with the Poynting - Thompson model and do a comparison.
All values of the retardation time attached to the specific viscoelastic model under the creep test are gathered in a set called *retardation spectrum*.
The following example demonstrates how time retardation times are computed and how those retardation time values are tied up to the roots of the characteristic equation related to the governing constitutive equation.

Example: Creep test and retardation time
The authors in [37] claim that the retardation times are equal

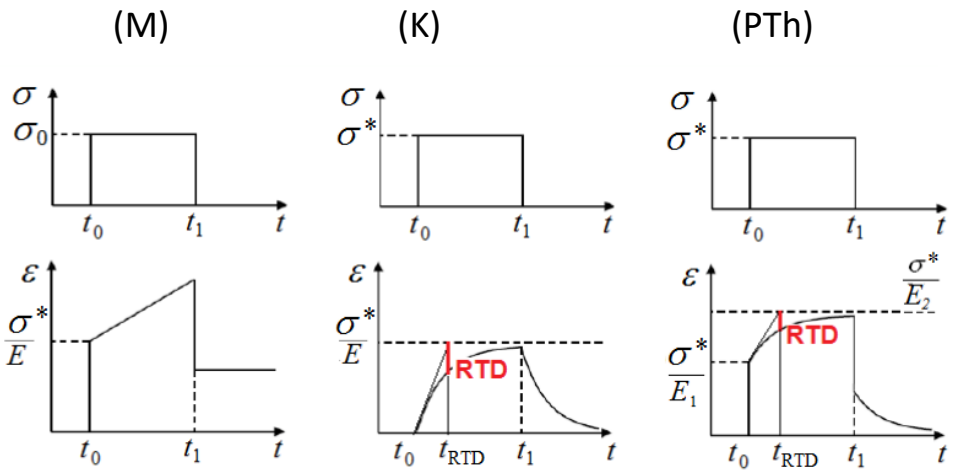

Figure 3.4: Maxwell, Kelvin-Voigt and Poynting - Thompson models subject to the creep test with cease. The retardation *RTD* is emphasized in red; the retardation time is equal to $t_{RTD} - t_0$.

to the negative reciprocal values to the characteristic equation roots. Indeed

$$t_{RTD} = \begin{cases} -\frac{1}{\lambda_C} & if \quad \lambda_C \neq 0, \\ 0 & if \quad \lambda_C = 0 \end{cases} \tag{3.60}$$

and

$$t_{RLX} = \begin{cases} -\frac{1}{\lambda_R} & if \quad \lambda_R \neq 0, \\ 0 & if \quad \lambda_R = 0, \end{cases} \tag{3.61}$$

where λ_C and λ_R are the roots of the characteristic equations within the creep and relaxation tests respectively.
It is necessary to mention herein, that besides the possible occurrence of zero values among the retardation times, there can occur some singularities arisen in the case of the relaxation. Whenever a submodel (S_0) is connected in parallel with a single viscous element: $(P) = (S_0)|(N)$, due to damping of (N), it is not possible to impose a non-zero instantaneous strain on such a model; theoretically infinite magnitude of force would be needed in order to impose any immediate non-zero strain. The tangent line slope is infinite and we say that the relaxation time does not exist. For $(S_0) = (H)$ in (P) we have (K), i.e. the

simplest (P). Its singularity accompanying the relaxation test, is performed in Figure 3.5. Apparently, since the (K) cannot undergo the relaxation test at all, there is no element in $RTD_{(K)}$.
Mathematically speaking - though the constitutive equation of (K), is a differential equation of the first order, only the time derivative of ε, and no derivative of σ appears there therein. Accordingly, if we supply the constant ε^* for $\varepsilon(t)$, we get the plain (not a differential) equation $\sigma = E\varepsilon^*$. We conclude that imposed to the (theoretical) relaxation test, no value of relaxation time is yielded. However, we put one "-" (empty place) inside the relaxation spectrum set for the sake of the emphasizing the fact that the constitutive equation of the model is a differential equation of the first order. And, let us summarizing finally. Whenever (K) or (P) is involved in a more complex viscoelastic model under relaxation test, one or more "-" appear in the relaxation spectrum set.
The following considerations are aimed to show how it works for a simple viscoelastic model. Subjecting the Kelvin-Voigt model (K), governed by its constitutive relation (3.15) to a creep test with a constant permanent stress load σ^* we definitely obtain the constitutive equation

$$\frac{\sigma^*}{\eta} = \dot{\varepsilon} + \frac{E}{\eta}\varepsilon \tag{3.62}$$

which is a linear differential ordinary differential equation with the constant coefficients. During the process of solving (3.62) we go through corresponding characteristic equation which is now of the form

$$\lambda_C + \frac{E}{\eta} = 0. \tag{3.63}$$

The unique root of (3.63) is $\lambda_C = -\frac{E}{\eta}\,[s^{-1}]$. Its physical meaning is that $-\frac{1}{\lambda_C}$ is exactly equal to the retardation time of (K) under the creep test. Well, let us proceed with the aim of proving this relation. Supposing the zero initial deformation we have the solution of (3.62) is of the form

$$\varepsilon(t) = \frac{\sigma^*}{E}\left(1 - e^{-\frac{E}{\eta}t}\right). \tag{3.64}$$

It is the creep function of Kelvin-Voigt model under the creep test. In line with the geometric interpretation of retardation, we would like to find the tangent line to this creep function at the initial point. So we differentiate (3.64) with respect to time, getting the creep rate function $\dot{\varepsilon}(t) = \frac{\sigma^*}{\eta} e^{-\frac{E}{\eta}t}$ of (K); and by evaluating its value at time instant $t_0 = 0$ we get the initial creep velocity

$$\dot{\varepsilon}(0) = \frac{\sigma^*}{\eta}. \tag{3.65}$$

Finally, by using the kinematic equation for the *straight line uniform motion time* = path/speed we derive the retardation time

$$t_{RTD} = \frac{\eta}{E} = -\frac{1}{\lambda_C}. \tag{3.66}$$

Let us additionally make a geometric confrontation with the aid of the Figure 3.4. We have led the tangent line to the creep function at the point $t_0 = 0$. Its slope is the derivative of the function $\varepsilon(t)$ in this point, (3.65). The intersection of the tangent line with the constant function, the horizontal asymptote of the creep function is a point with coordinates t_{RTD} and $\frac{\sigma^*}{E}$.
The *retardation spectrum* $RTD_{(model)}$ is a set containing all retardation time values. The number of members (non-zero and zero) in the time retardation spectrum set is equal to the number of irreducible pistons involved *in parallel*, in the viscoelastic model. As it was performed above, all non-zero retardation time values are yielded directly form the characteristic equation arisen from the constitutional equation by applying the creep test. They are the negative reciprocal values of the non-zero roots of this characteristic equation. On the other hand, the zero retardation time arises from the serial connection of (N) herein when no retardation appears. It is so in the case of the Maxwell model, see Figure 3.4, left.
We can handle the Poynting-Thompson model (Figure 3.3) similarly as the Kelvin-Voigt model, yielding one retardation time value $\left\{\frac{\eta}{E_2}\right\}$.

$$RTD_{(M)} = \{0\},$$
$$RTD_{(K)} = \left\{\frac{\eta}{E}\right\},$$

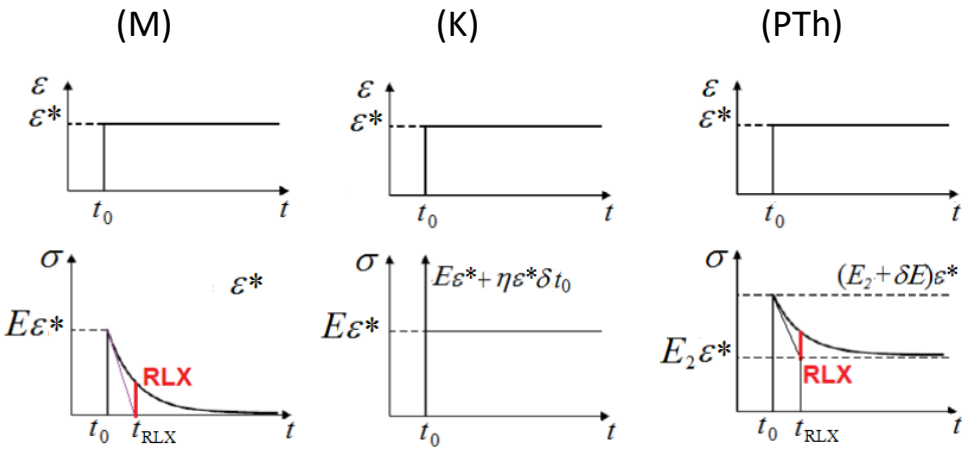

Figure 3.5: Relaxation test carried out on the Maxwell, the Kelvin-Voigt (just theoretically - singularity occurs there) and the Poynting - Thompson models, relaxation *RLX* is emphasised in red, relaxation time=$t_{RLX} - t_0$.

$$RTD_{(PTh)} = \left\{ \frac{\eta}{E_2} \right\},$$

$$RTD_{(S)} = \left\{ \frac{\eta_1}{E_1}, 0 \right\}.$$

Example: Relaxation test and relaxation time
During a relaxation test, we get the *relaxation time spectrum* in the time scale. We trace the decrease of the stress function as the response to the instantly imposed constant permanent deformation ε^*.
Let us take (*M*) and subject it to the relaxation test. Initially, due to an immediate deformation, the stress immediately increases to its maximum, and at the same time, it starts its slower and slower decrease down towards zero or another reference value reflecting the type of the model.
The relaxation time is measured on (*M*) from the time moment when the load begins to act, up to the instant, when the virtual

stress drop (downwards linearly in the direction of the initial stress velocity vector) reaches zero or corresponding reference value reflecting the configuration of the model, compare (*M*) and (*PTh*) models in Figure 3.5. Check also the geometrical interpretation of the situation with relaxation and relaxation time t_{RLX}. The endpoint of the relaxation time interval is measured in the intersection of that tangent line to the relaxation curve in the load/response initiation time, with the abscissa if we refer to (*M*), or with a reference horizontal line, if we consider a more complex, e.g. Poynting-Thompson model.
Analogously to the previous example, by substituting ε^* for $\varepsilon(t)$ in (3.20) yielding the characteristic equation

$$0 = \frac{\lambda}{E} + \frac{1}{\eta}, \tag{3.67}$$

we get the unique relaxation time; and the relaxation time spectrum $\left\{\frac{\eta}{E}\right\}$ - the one element set for the Maxwell model. The same procedure yields $\left\{\frac{\eta}{E_2}\right\}$ - the relaxation time spectrum for the Poynting - Thompson model involving one dash-pot only. For each non-zero λ_R, the root of characteristic equation of any model, the relationship between the relaxation time t_R and the corresponding characteristic equation root λ_R is

$$t_{RLX} = -\frac{1}{\lambda_R} \tag{3.68}$$

The number of elements in the time relaxation spectrum is equal to the number of irreducible pistons involved *in series* in the viscoelastic model.
In reality, it is impossible to impose an immediate non-zero strain (step-wisely) on the Kelvin Voight model. The force of infinitesimal magnitude would be necessary. We say that (*K*) does not relax and we put a dash in relaxation time spectrum pointing out to the present singularity herein We can briefly summarise:

$$RLX_{(M)} = \left\{\frac{\eta}{E}\right\},$$

$$RLX_{(K)} = \{-\},$$

$$RLX_{(PTh)} = \left\{ \frac{\eta}{E_2} \right\},$$

$$RLX_{(S)} = \{-,-\},$$

etc. By the same procedure we can calculate spektra of retardation and relaxation times for any viscoelastic model. For more information see [53].

3.2.5 Viscoelastic modelling application Human plantar aponeurosis (fascia)

The human body consists of typical rheological components exclusively. The reason is the presence of water in each cell of the body. Various fibrous connective tissues, i.e. tendons and ligaments are typical viscoelastic materials. Even body liquids (blood, lymph) perform more or less rheological behaviour. We will next focus on the flat tendon, which is located at the very bottom of the human foot, under the plantar arch, and which plays an important role in walking, running, etc. It is called the plantar aponeurosis or fascia [41, 48]. In Figure 3.6, the anatomy of aponeurosis, its mechanical functioning and a simplified viscoelastic model are illustrated. Admitting the appropriate level of simplification, and regarding the expected (physiological and non destructive) range of load, for a short time period of load, we have chosen the Burgers model
$(aponeurosis) = (H) - (K) - (N) = (H) - [(H)|(N)] - (N)$.
By using the tools of Chapter 3.2 first we are supposed to derive the global *constitutive relation* of the model performing, simulating the *fascia*. When handling a more complex viscoelastic model, in order to determine the constitutive relation, it is more convenient to deal with the model regarded as the parallel or serial connection of several simpler submodels, (see the model split of aponeurosis together with physical parameters in Figure 3.6, right). The model is regarded as a serial connection of three components: Elastic body - block 1, Kelvin-Voigt submodel - block 2, Newton body - block 3. Indeed, we will use the formula $(H_1) - [(H_2)(N_2)] - (N_3)$, where indices set out the incidence with the involving blocks.

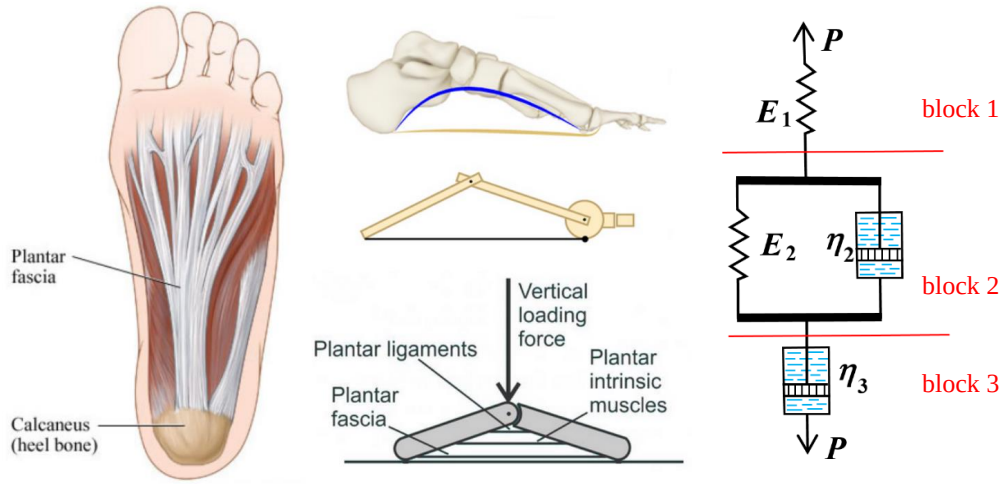

Figure 3.6: Plantar human aponeurosis (fascia); Anatomy (left), mechanical models (middle), simplified viscoelastic model (right) [55, 29].

Due to the serial connection of three blocks, we can write down the global conditional stiffness coefficient directly, see (3.27):

$$\frac{1}{\hat{E}} = \frac{1}{E_{elast}} + \frac{1}{E_K} + \frac{1}{E_{visc}}, \tag{3.69}$$

where the particular conditional stiffness values come from the particular blocks:

- block 1. $\sigma_1 = E_1\varepsilon_1 \Rightarrow E_{elast} = E_1$
- block 2. $\sigma_2 = (E_2 + \eta_2 D)\varepsilon \Rightarrow E_K = E_2 + \eta_2 D$
- block 3. $\sigma_3 = \eta_3 D\varepsilon \Rightarrow E_{visc} = \eta_3 D$

The fact that while the conditional stiffness coefficient E_{elast} stands as a purely physical parameter, E_K and E_{visc} are differential operators since they couple the physical properties of the elementary members with the time derivative operator. By substituting these conditional stiffness coefficients into (3.69), we

obtain the resulting conditional stiffness of the aponeurosis in the form

$$\hat{E} = \frac{E_1 E_2 \eta_3 D + E_1 \eta_2 \eta_3 D^2}{E_2 \eta_3 D + \eta_2 \eta_3 D^2 + E_1 \eta_3 D + E_1 E_2 + E_1 \eta_2 D}. \tag{3.70}$$

Thereafter, in the sense of (3.25), the stress-strain relation, the mechanical behaviour governing equation of human foot fascia written in implicit form is

$$\eta_2 \eta_3 \ddot{\sigma} + (E_2 \eta_3 + E_1 \eta_3 + E_1 \eta_2)\dot{\sigma} + E_1 E_2 \sigma = \\ = E_1 \eta_2 \eta_3 \ddot{\varepsilon} + E_1 E_2 \eta_3 \dot{\varepsilon}. \tag{3.71}$$

From now on, having the constitutive relation (3.71), we are capable of investigating the *mechanical response* of the matter standing beyond the model theoretically, subjected to any action load. Let us execute both standard tests described in Chapter 3.1.2.

Creep test with cease carried out on aponeurosis viscoelastic model

The constant tension stress $\sigma^* = F_G/A_{plant}$ $[Pa]$ corresponding to the half weight of the body in an unmoving standing posture is imposed on the initially motionless plantar fascia at the time instant $t_0 = 0$. The size of the plantar aponeurosis area of both feet herein is denoted as A_{plant} $[cm^2]$.
We assume that the healthy plantar tissue for this σ^* is kept within a suitably safe range of load in which no injury incident occurs. Consequently, if we use the constant value σ^* as $\sigma(t)$ in (3.71), we get a linear ordinary differential equation with constant coefficients and with non-zero right hand side

$$E_1 \eta_2 \eta_3 \ddot{\varepsilon} + E_1 E_2 \eta_3 \dot{\varepsilon} = E_1 E_2 . \sigma^* \tag{3.72}$$

Equation (3.72) governs the process of deformation of aponeurosis represented by its viscoelastic model over time during the creep test. It is a linear 2^{nd}-order ordinary differential equation with constant coefficients, and two additional conditions are required to make the problem uniquely solvable. The two

required conditions can be the initial strain value and the initial strain rate value. All these initial conditions can be specified block by block. Having the aponeurosis viscoelastic model split into three particular blocks, connected *in series* as outlined in Figure 3.6: block 1 = (H_1), block 2 = $(K) = (H_2)|(N_2)$, block 3 = (N_3). Then the geometric equations of the connection in series, in accordance with (2.17),

$$\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t). \tag{3.73}$$

Due to the solitaire Hookean member connected *in series*, performed as block 1 in Figure 3.6 right, the immediate initial strain reaction to the immediate stress action at time instant $t_0 = 0$ is just upon it and it appears immediately:

$$\varepsilon(0) = \varepsilon_1(0)\frac{\sigma^*}{E_1}. \tag{3.74}$$

It is worth recalling that as far as a stepwise constant action function is concerned, block 1 is responsible for all immediate reaction changes, and jumps, and only for them. But it does not contribute to the movement between those jump points. On the other hand, blocks 2 and 3, remain unmoved in the moment of a jump due to the involved damping member and no strain occurs on them. But they start to move at the same moment and evolve the responding movement function at the very next moment. Mathematically, it means that both these blocks are decisive when deriving the initial deformation rate $\dot{\varepsilon}$. As was already mentioned, the total deformation of aponeurosis represented by this model is summed up from the deformations of all (three) particular blocks at any time instant $t \geq t_0$. Block 1 does not contribute to the movement between those jump points time instants. Obviously, the total deformation rate comes directly from (3.73), by its differentiating it with respect to the time variable:

$$\dot{\varepsilon}(t) = \dot{\varepsilon}_1(t) + \dot{\varepsilon}_2(t) + \dot{\varepsilon}_3(t). \tag{3.75}$$

Let us have a look at the strain functions of the particular blocks of aponeurosis viscoelastic model. Indeed we want to specify the individual addends in (3.75). Under the constant persisting

load σ^* imposed on the tissue represented by the viscoelastic model, the constitutive relations of three blocks are of the form

$$\begin{aligned} \sigma_1 &= E_1\varepsilon_1(t), \\ \sigma_2 &= E_2\varepsilon_2(t) + \eta_2\dot{\varepsilon}_2(t), \\ \sigma_3 &= \eta_3\dot{\varepsilon}_3(t). \end{aligned} \tag{3.76}$$

Since the geometric equations of the connection in series of

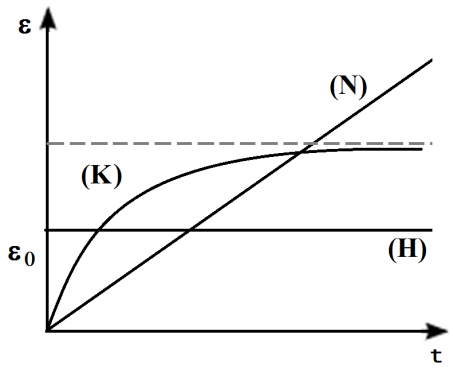

Figure 3.7: Strain functions of individual members of the human plantar aponeurosis viscoelastic model subjected to a constant load; summing them up yields the resulting creep function.

blocks 1, 2 and 3 following from (2.17) ensure an equality of particular stress functions of three particular members

$$\sigma^* = \sigma_1(t) = \sigma_2(t) = \sigma_3(t), \tag{3.77}$$

the physical equations of three blocks build up a system of three equations with three unknown strain functions

$$\begin{aligned} \sigma^* &= E_1\varepsilon_1(t), \\ \sigma^* &= E_2\varepsilon_2(t) + \eta_2\dot{\varepsilon}(t), \\ \sigma^* &= \eta_3\dot{\varepsilon}_3(t). \end{aligned} \tag{3.78}$$

The second and the third equations of (3.78) are first-order linear differential equations. Accordingly, for the sake of ensuring the uniqueness of the solution, for each of both equations, one additional condition is needed. Indeed, we have the particular initial conditions $\varepsilon_2(0) = 0$ and $\varepsilon_3(0) = 0$ justified above.

The time-dependent strain function $\varepsilon_2(t)$ of block 2 is given as a solution to the second equation in (3.78), the first-order linear differential equation. Together with the attached zero initial condition $\varepsilon_2(0) = 0$ we get the unique solution

$$\varepsilon_2(t) = \frac{\sigma^*}{E_2}\left(1 - e^{-\frac{E_2}{\eta_2}t}\right), \tag{3.79}$$

see the method of variation of parameters in Chapter 3.2.1. When we differentiate (3.79) with respect to time, we get the strain rate of block 2: $\dot{\varepsilon}_2(t) = \frac{\sigma^*}{\eta_2}e^{-\frac{E_2}{\eta_2}t}$. It is worth recalling that behind the zero strain rate of block 1, the strain rate function of block 2 changes in time while the strain rate function of the third block is constant. The last one: $\dot{\varepsilon}_3(t) = \frac{\sigma_3}{\eta_3} = \frac{\sigma^*}{\eta_3}$ which follows directly from the third equation of the system (3.78). Finally, we just figure up the sum (3.75)

$$\dot{\varepsilon}(t) = 0 + \frac{\sigma^*}{\eta_2}e^{-\frac{E_2}{\eta_2}t} + \frac{\sigma^*}{\eta_3}, \tag{3.80}$$

yielding the global strain rate function. Its initial required value at $t_0 = 0$ is

$$\dot{\varepsilon}(0) = \sigma^*\left(\frac{1}{\eta_2} + \frac{1}{\eta_3}\right). \tag{3.81}$$

In a physiologic case, (H_1) is able to bear and store all receiving initial energy imposed on the plantar fascia tissue and E_1 is not overloaded. The (*aponeurosis*) potential energy immediately stored by (H_1) is initially ready to be used for a prospective reverting process. But, with the lapse of time, since the load persists, the energy is continually dissipated, resulting in the loss of both the potential energy and the recovery capability. It means a certain permanent deformation appears and persists even a long time after the load cease. Nevertheless, there are physiological processes inside the living human body that usually neutralize this permanent strain impact - partially or totally. A more detailed exploration concerning energy storage and dissipation is provided in the next chapter.
We shortly conclude that the model (*aponeurosis*) fits satisfactory to the human plantar aponeurosis for a short period of time

and without any overburdening during the mechanical loading. These two limitations exist because the parental organism is still alive, and many physiological processes run in synergy.
Accordingly, many other factors influence the process of aponeurosis mechanical behaviour. Now it is worth emphasizing that, as far as mathematical treatment is concerned, the use of equation (3.71) as the stress-strain relation of the human plantar fascia is justified.
At last, the Cauchy initial value problem (3.72), (3.74), (3.81) can be solved by using standard methods of ordinary differential equations, yielding the unique solution, the function operating within the loading period $\langle t_0, t_1\rangle$, let us denote it as $\varepsilon_L(t)$:

$$\varepsilon_L(t) = \sigma^* \left(\frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{E_2} e^{-\frac{E_2}{\eta_2} t} + \frac{t}{\eta_3} \right). \tag{3.82}$$

The creep function governs the movement. It is valid while the load remains unchanged.
Naturally, we next extend the creep test by a stepwise cease of load simulating walking, running or jumping. The stepwise action function is assumed, e.g. $\sigma^* \neq 0$ in $\langle t_0, t_1\rangle$ and $\sigma^* = 0$ for $t > t_1$. In such a case, the resulting decrease in the creep function is of interest.
After the immediate (stepped) cease of the load at time instant t_1, the creep function changes instantaneously. The non-smoothness in the still continuous creep function appears at the time instant t_1. Its new leg $\varepsilon_U(t)$ (index U stands for "unloading") can be computed from the new initial problem consisting of the same governing differential equation (3.78), with a new initial condition, the current values of $\varepsilon_2(t_1) = \frac{\sigma^*}{E_2}(1 - e^{-\frac{E_2}{\eta_2} t_1})$ and $\varepsilon_3(t_1) = \frac{\sigma^*}{\eta_3} t_1$, where ε_3 follows from the third equation of (3.78) and the zero initial condition at $t_0 = 0$.

$$\begin{aligned} \varepsilon_2(t) &= \frac{\sigma^*}{E_2}\left(e^{\frac{E_2}{\eta_2}(t_1 - t)} - e^{-\frac{E_2}{\eta_2} t} \right), \quad t \geq t_1, \\ \varepsilon_3(t) &= \frac{\sigma^*}{\eta_3} t_1, \quad t \geq t_1. \end{aligned} \tag{3.83}$$

Having the strain values in the cease instant t_1: $\varepsilon_2(t_1)$ and $\varepsilon_3(t_1)$, operating as the initial conditions in the second and the

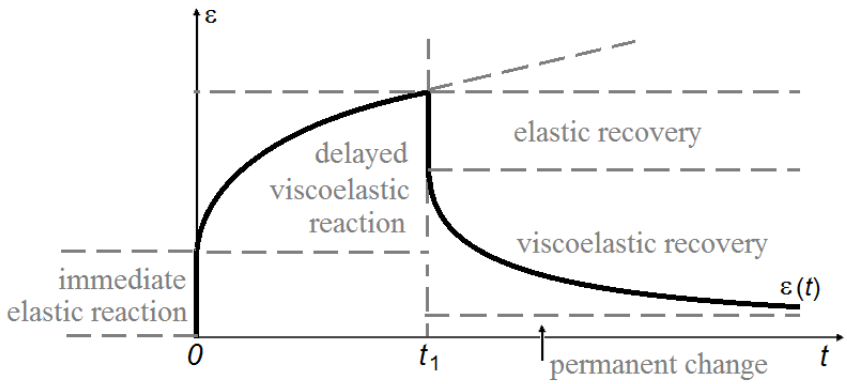

Figure 3.8: Human plantar aponeurosis under the creep test with cease. Viscoelastic deformation as a response to a constant stress maintained within the time interval $\langle 0, t_1\rangle$, ceased immediately at time instant t_1.

third equations of (3.78) over the time period $t \geq t_1$ and recalling $\varepsilon_1(t_1) = \frac{\sigma^*}{E_1}$, following straightly from the first equation of (3.78), we can complete the "unload leg" of the total strain function as the sum (3.73):

$$\varepsilon_U(t) = \frac{\sigma^*}{E_2}\left(e^{\frac{E_2}{\eta_2}(t_1-t)} - e^{-\frac{E_2}{\eta_2}t}\right) + \frac{\sigma^*}{\eta_3}t_1, \quad t \geq t_1. \tag{3.84}$$

In Figure 3.8 we see that after the peak point $\varepsilon_L(t_1)$ where the nonsmoothness of $\varepsilon(t)$ appears, the solution continues with its second decreasing leg. Finally, we can summarize that for the stepwise stress function

$$\sigma(t) = \begin{cases} \sigma^* & 0 \leq t \leq t_1, \\ 0 & t > t_1. \end{cases}$$

as the action, the reaction as a nonsmooth, but still continuous creep function

$$\varepsilon(t) = \begin{cases} \varepsilon_L & 0 \leq t \leq t_1, \\ \varepsilon_U & t > t_1. \end{cases} \tag{3.85}$$

The graphical performance of the creep function (3.85) can be seen in Figure 3.8. Its hysteretic behaviour is illustrated in Figure 3.9.

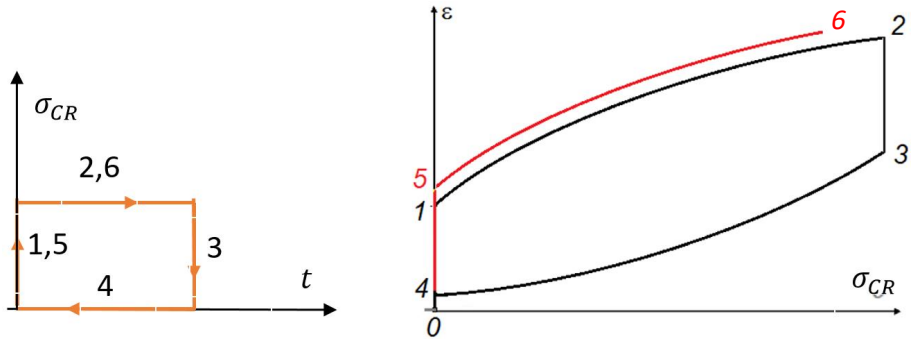

Figure 3.9: Hysteretic behaviour of the human plantar aponeurosis tissue (right) under the stepwise load simulating walking (left) [32].

Relaxation test carried out on viscoelastic model of human plantar aponeurosis

The constant deformation load $\varepsilon(t) = \varepsilon^*$ is imposed at time instant t_0, and kept in time, to the aponeurosis represented by Burger's viscoelastic model. Likewise as in the case of the creep test, we carry out our analysis with both previous assumptions: a "short" period taken into account and just a physiological range of load is supposed, under which no destructive change of the tissue can occur. Then the following considerations can be pursued.
The procedure for deriving the relaxation function $\sigma(t)$ is of a similar spirit as in the case of the creep test. Here again we need to use the split of the aponeurosis viscoelastic model to those three particular blocks, connected *in series*. Though the deformation of the entire model ε^* remains the same alongside the test, the particular deformations of particular blocks change over time. It follows from (3.73) that the total deformation, kept unchanged within the duration of the whole theoretical experiment $\varepsilon(t) = \varepsilon^*$, is composed from three addends that change in time

$$\varepsilon^* = \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t). \tag{3.86}$$

In order to calculate the responding global stress (relaxation

function), we will need to evaluate each of these three addends one by one. We can do this as follows.
The governing differential equation arises from the constitutive relation (3.71) with $\varepsilon(t) = \varepsilon^*$. This bears

$$\eta_2\eta_3\ddot{\sigma} + (E_2\eta_3 + E_1\eta_3 + E_1\eta_2)\dot{\sigma} + E_1E_2\sigma = 0, \tag{3.87}$$

a homogeneous differential equation with constant coefficients that governs the relaxation process of the aponeurosis under a constant deformation. The equation is easily solvable with appropriate initial conditions. But we do not have them directly.
So, let us first think a bit physically about the initial situation, i.e. the situation with the model at time instant t_0. Blocks 2 and 3 are disabled to bear the immediate strain. In the case of block 2, the viscous member connected *in parallel* causes the damping effect, and block 3 is a viscous member that always damps itself. That is why all the action deformation imposed on the model is immediately carried by block 1, the elastic component connected *in series* to the rest of the damped model can do that; hence $\varepsilon^* = \varepsilon_1(0)$. Indeed, we can conclude from the previous consideration (having the physical equation of (H) in mind) that $\sigma(0) = \sigma_1(0) = \varepsilon^* E_1$.
For deriving the relaxation function rate, we have to take (3.86) for $n = 3$ and the equality of all the component (blocks) stress values connected *in series* $\sigma(t) = \sigma_1(t) = \sigma_2(t) = \sigma_3(t)$, see (2.17), into account, together with appropriate physical equations of the three blocks of the model

$$\left[\sigma_1(t) = \right]\sigma(t) = E_1\varepsilon_1(t), \tag{3.88}$$

$$\left[\sigma_2(t) = \right]\sigma(t) = E_2\varepsilon_2(t) + \eta_2\dot{\varepsilon}_2(t), \tag{3.89}$$

$$\left[\sigma_3(t) = \right]\sigma(t) = \eta_3\dot{\varepsilon}_3(t). \tag{3.90}$$

System [(3.86), (3.88) - (3.90)] couples one linear equation and two linear differential equations. Together with the initial conditions

$$\begin{aligned} \varepsilon_1(0) &= \varepsilon^* \\ \varepsilon_2(0) &= 0 \\ \varepsilon_3(0) &= 0 \end{aligned} \tag{3.91}$$

that were focused on and reasoned above, the system can be solved by using standard mathematical tools. All coefficients

in all equations of the system are constant; hence the analytical solution is reasonable. However, the resulting deformation functions $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and the relaxation function are too complicated when written down generally. Thus, only the graphical interpretation of the particular strain functions is provided, namely:

* Figure 3.10 performs the deformation of the spring - elastic member (H_1) linked *in series* to the rest of the model during the constant total deformation, the action load ε^* of relaxation test.

* Figure 3.11 performs the damped deformation of Kelvin-Voigt submodel of aponeurosis viscoelastic model under the constant total deformation ε^*.

* Figure 3.12 performs the deformation of the dash-pot connected *in series* to the rest of the aponeurosis model during maintained constant deformation ε^* load. It can be seen that deformation continues increasing linearly until the load persists.

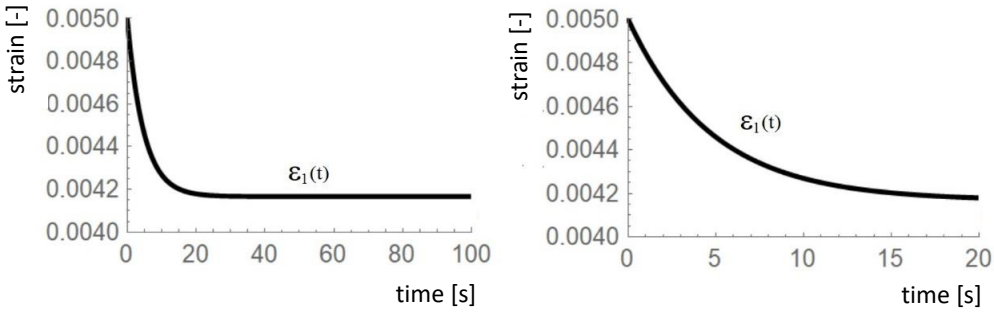

Figure 3.10: Responding strain function $\varepsilon_1(t)$, specific deformation of block 1, the Hookean member (H_1) of the viscoelastic model aponeurosis under the relaxation test; larger scale (left), smaller scale (right).

The total strain decomposition (3.86) is graphically interpreted in Figure 3.13.

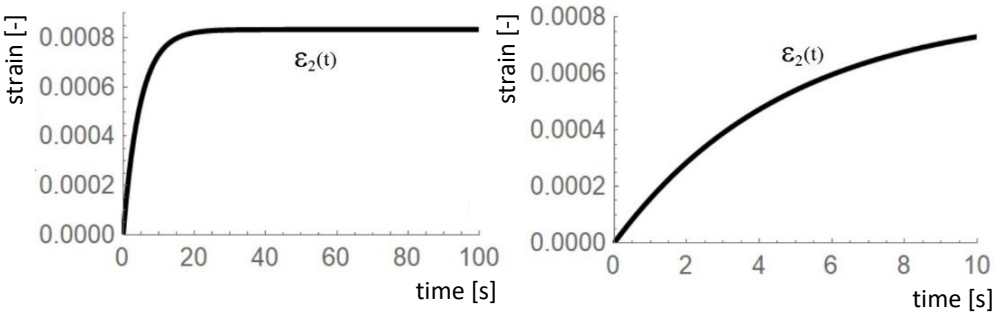

Figure 3.11: Responding partial strain function of block 2, the Kelvin-Voigt submodel ($H_2|N_2$) of the aponeurosis viscoelastic model under the relaxation test, i.e. $\varepsilon_2(t)$ in larger scale (left), smaller scale (right).

3.3 Energy storage and dissipation

This chapter deals with the energy of a viscoelastic body. As has been mentioned above, in the case of the the loading and unloading process of a viscoelastic material, the energy dissipation phenomenon occurs due to the presence of one or more viscous elements that do not store any energy. This released heat-dissipated energy usually turns into heat. When the amount of heat is "too big", the ambient temperature can increase significantly, so far as the physical parameters are affected. In such a case, linearity is not kept any more, as well as the Boltzmann superposition principle is no longer valid; and the non-linear approach has to be employed. From the theory of thermodynamics, we know that the change of the inner energy of a viscoelastic material is induced either by a work-performance or by the exchange of heat. The first law of thermodynamics, the balance equation in a closed system is mostly referred to when the thermodynamic consistency requirement of the rheological model is mentioned. However, we start with linear modelling and concurrently check the amount of released heat, quantify the temperature increase and verify the change in the physical parameters.
Accordingly, from now on, within the scope of the isothermal

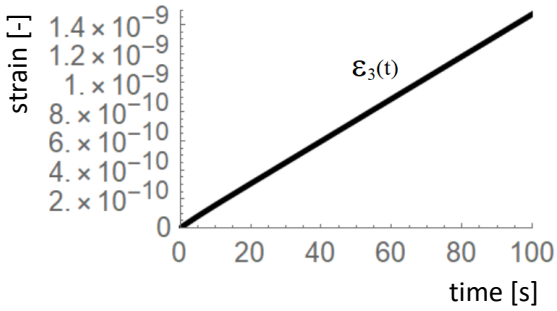

Figure 3.12: Responding strain function ε_3, the specific deformation of block3, the Newton member (N_3) of the viscoelastic model of human plantar aponeurosis under the relaxation test; larger scale (left), smaller scale (right).

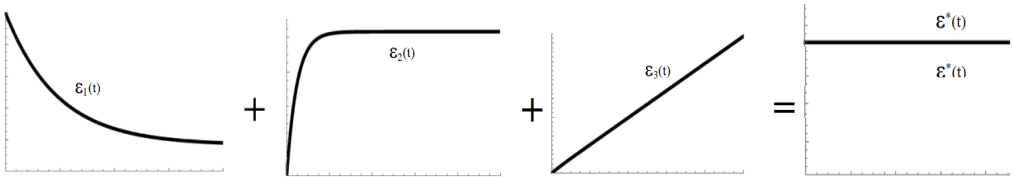

Figure 3.13: Illustration of the constant total deformation function ε^* split to the sum of the strain functions of particular blocks during the relaxation test.

viscoelastic investigation, we can suppose that the mechanical energy received by the viscoelastic model is either stored or dissipated, $U = U_p + U_d$. Subject to a load, the one degree of freedom viscoelastic models' behaviour is observed, together with the observation of the energy – total, stored and dissipated. Nevertheless, the only stored energy in the viscoelastic model is the potential energy. The ratio of the dissipated to the total energy can further stand as a measure of the reversibility of the model.

It is worth saying here that if the dash-pot is connected in series at the end of a model, as it is e.g. in the case of the Maxwell or the Schofield models, check in Figures (2.9) and (3.3), the model is irreversible. Each rheological element, as well as a

viscoelastic model, governed by its corresponding constitutive relation with the potential energy $U_p \geq 0$, has to be thermodynamically consistent. Hence, the energy dissipation rate of the model has to be non-negative, [20]. Now, let us trace the total deformation energy flow during a load process and then explore its partition into the potential energy and the dissipated energy components, having the constitutive equation of the viscoelastic model, which expresses the dependence of the reaction on the action, still in mind. First, we will focus on the total work of deformation quantification. The stress power of the viscoelastic model, i.e. the rate of the deformation work quantified per time unit and volume unit, is, [47]

$$\dot{U} = \sigma(t)\dot{\varepsilon}(t). \tag{3.92}$$

Consequently, by integration of (3.92), the deformation work itself can be expressed and quantified per time interval $\langle 0, t\rangle$; at the same time the initial value $U(0)$ of the energy is taken into account as well:

$$U(t) = U(0) + \int_0^t \dot{U}(a)\mathrm{d}a = U(0) + \int_0^t \sigma(a)\dot{\varepsilon}(a)\mathrm{d}a. \tag{3.93}$$

The initial instantaneous energy $U(0)$ is always non-negative - mostly impulsively increased or decreased within one moment or zeroed in some cases. Further explanation about this is given below in this chapter. Each of the expressions (3.92) and (3.93) involves both stress and strain functions. However, we should bear in mind that typically, in the continuum mechanics tasks, there is an action imposed on a body, and the reaction induced is focused, and, usually, either the stress is an action and the strain is the reaction or vice versa. Hence, for a more convenient subsequent utilization, it is worth expressing (3.92) and (3.93) explicitly either in the sense of stress exclusively or in the sense of strain exclusively. For this purpose, we can employ the time-dependent material characteristics: the relaxation modulus $\tilde{E}(t)$ and the compliance modulus $\tilde{C}(t)$, which are specific for particular load type and a particular viscoelastic model, see Chapter 3.1.2. This way, we can switch between the stress and the strain - according to the actual need. After all, the deformation energy rate (3.92) written either in the sense of strain

exclusively or in the sense of the stress exclusively acquires the form

$$\dot{U}(t) = \tilde{E}(t)\varepsilon(t)\dot{\varepsilon}(t) \tag{3.94}$$

or

$$\dot{U}(t) = \sigma(t)\frac{\mathrm{d}}{\mathrm{d}t}\Big(\tilde{C}(t)\sigma(t)\Big) \tag{3.95}$$

and consequently, the total deformation energy (3.93) formulated either in the sense of strain exclusively is

$$U(t) = U(0) + \int_0^t \tilde{E}(a)\varepsilon(a)\dot{\varepsilon}(a)\mathrm{d}a \tag{3.96}$$

or in the sense of stress exclusively

$$U(t) = U(0) + \int_0^t \sigma(a)\frac{\mathrm{d}}{\mathrm{d}a}\Big(\tilde{C}(a)\sigma(a)\Big)\mathrm{d}a. \tag{3.97}$$

Remark
General rules for the derivation of the time-dependent material characteristics $\tilde{E}(t)$ and $\tilde{C}(t)$ are explored in [25], and the synopsis is provided in [24]. Each of these functions of time links the configuration of the viscoelastic model with all physical equations of all included members. It is always the ratio of action and reaction that varies in time. Specific cases of time-dependent material characteristics $\tilde{E}(t) = \frac{\sigma(t)}{\varepsilon(t)}$ and $\tilde{C}(t) = \frac{\varepsilon(t)}{\sigma(t)}$ derivation for the Maxwell model, subjected to creep and relaxation tests, are dealt with in Chapter (3.1.2).

Roughly speaking, due to the presence of elastic elements in viscoelastic models, a part of the deformation energy can be stored by the elastic members, while the rest of this energy is dissipated thanks to the viscous elements present therein. The stored energy is later, e.g. after the load ceases, utilized for a partial or total deformation reversal of the viscoelastic body. The dissipated energy is usually converted into the heat and spread irrecoverably into the ambient. The energy dissipation rate through the dashpot (N)

$$\dot{U}_d = \sigma_N(t)\dot{\varepsilon}_N(t) \tag{3.98}$$

in the sense of strain and the sense of stress, respectively, acquires the form:

$$\dot{U}_d = \eta \dot{\varepsilon}_N^2(t), \tag{3.99}$$

$$\dot{U}_d = \frac{1}{\eta}\sigma_N^2(t). \tag{3.100}$$

By integrating (3.98), $U_d(t) = U_d(0) + \int_0^t \dot{U}_d(a)\mathrm{d}a$, i.e. the amount of the energy dissipated within the period $\langle 0, t\rangle$ of the mechanical loading process can be quantified per volume unit either in the sense of strain or in the sense of stress, respectively:

$$U_d(t) = U_d(0) + \int_0^t \eta \dot{\varepsilon}_N^2 E(a)\mathrm{d}a, \tag{3.101}$$

$$U_d(t) = U_d(0) + \int_0^t \frac{1}{\eta}\sigma_N^2(a)\mathrm{d}a. \tag{3.102}$$

Definitely, with respect to (3.98), the physical relation of Newton's viscous matter (2.11) stands as the switch tool between (3.99) and (3.100), and between (3.101) and (3.102).
Next, we can proceed similarly with exploring the potential energy. Let us start with the potential (stored by (H)) energy rate $\dot{U}_p = \sigma_H(t)\dot{\varepsilon}_H(t)$ formulated either in the sense of the strain:

$$\dot{U}_p = E\varepsilon(t)\dot{\varepsilon}_H(t) \tag{3.103}$$

or in the sense of the stress:

$$\dot{U}_p = \frac{1}{E}\sigma_H(t)\dot{\sigma_H}(t). \tag{3.104}$$

Evidently, the potential energy itself is

$$U_p(t) = U_p(0) + \int_0^t \dot{U}_p(a)\mathrm{d}a.$$

Written in the sense of strain and stress, respectively, it is

$$U_p(t) = U_p(0) + \int_0^t E\varepsilon_H(a)\dot{\varepsilon}_H(a)\mathrm{d}a, \tag{3.105}$$

$$U_p(t) = U_p(0) + \int_0^t \frac{1}{E}\sigma_H(a)\dot{\sigma}_H(a)\mathrm{d}a. \tag{3.106}$$

Since a dash-pot prevents an immediate deformation, an action-reaction non-zero immediate initial deformation appears only in the case when there is an elastic element (H) connected in series to the rest of the model within the configuration of a viscoelastic model. In such a case, the initial dissipation energy is zero, while (H) stores all possible initial immediate potential energy

$$U_p(0) = \frac{1}{2}E\varepsilon_H^2(0) = \frac{1}{2E}\sigma_H^2(0). \tag{3.107}$$

Consequently, the stored energy can be determined

$$U_p(t) = \frac{1}{2}E\varepsilon_H^2(0) + \int_0^t E\varepsilon_H(a)\dot{\varepsilon}_H(a)\mathrm{d}a =$$

$$= \frac{1}{2E}\sigma_H^2(0) + \int_0^t \frac{1}{E}\sigma_H(a)\dot{\sigma}_H(a)\mathrm{d}a. \tag{3.108}$$

Both integrals in (3.105) and (3.106) can be also regarded (and evaluated) as Stieltjes integrals:

$$\int_0^t E\varepsilon_H(a)\dot{\varepsilon}_H(a)\mathrm{d}a = E\int_{\varepsilon(0)}^{\varepsilon(t)} \varepsilon_H(a)\mathrm{d}\varepsilon(a) = \frac{1}{2}E\Big(\varepsilon_H^2(t) - \varepsilon_H^2(0)\Big), \tag{3.109}$$

$$\int_0^t \frac{1}{E}\sigma_H(a)\dot{\sigma}_H(a)\mathrm{d}a = \frac{1}{E}\int_{\sigma_H(0)}^{\sigma_H(t)} \sigma_H(a)\mathrm{d}\sigma_H(a) = \frac{1}{2E}\Big(\sigma_H^2(t) - \sigma_H^2(0)\Big). \tag{3.110}$$

Having the Sieltjes integral at hand, we can derive the forms specifying the potential and the dissipated energy of viscoelastic models of various configurations.

If the magnitude of energy imposed on the medium represented by the viscoelastic model exceeds the storing capability of a spring connected to the rest of the viscoelastic model, the remaining energy is initially step-wisely dissipated and immediately after that initial jump, the amount of dissipated energy starts to decrease.

If there are more elastic or more viscous elements involved

in the viscoelastic model, then the corresponding energies are summed up adequately, with respect to the configuration of the model.
In Figure 3.14 some examples of energy performance are provided for the Maxwell model which is subject to several types of stress load - the action functions $A(t)$ that bring the strain response - the reaction $R(t)$. The deformation energy is performed in the second column of the table as partitioned into the potential and the dissipated one. For the sake of synopsis, a constant, a linear non-constant, an exponential and some periodical functions were chosen as actions. In the third column, the fluidity function is placed, denoted by $V(t)$. The fluidity function is equal to the ratio of the dissipated energy and the total energy of deformation. The hysteretic behaviour of models has to do a lot with this function. The dashed line functions in the third column can stand as a measure of the reversibility of the model. Other types of energies involved in the process of loading are neglected.
As far as the fluidity of the material is concerned, the relaxation time is worth recalling. The fluidity can be quantified by its length. From the proportionality $t_{rlx} = \frac{\eta}{E}$ the conclusion follows directly: The longer the relaxation time, the more "liquid-like" behaviour of the material, the shorter the relaxation time, the more "solid-like" behaviour of the material.

3.4 Structured viscoelastic models

Even before building up huge and heavy constructions like hydro-power stations, it is inevitable to gather as much information as possible about the soil beneath the construction. An appropriate geologic exploration always goes ahead in line with the theoretical investigations, finding out what kind of soil layers there are, and what their geometry and physical parameters are. If the geologic exploration reveals the shapely layered type of soil at the place where the future heavy construction is planned to be built on, a chain viscoelastic model is attached to it.
In Figure 3.15, an illustration of Slovak hydro-power station Gabčíkovo is provided, involving the photo from above, technical drawings

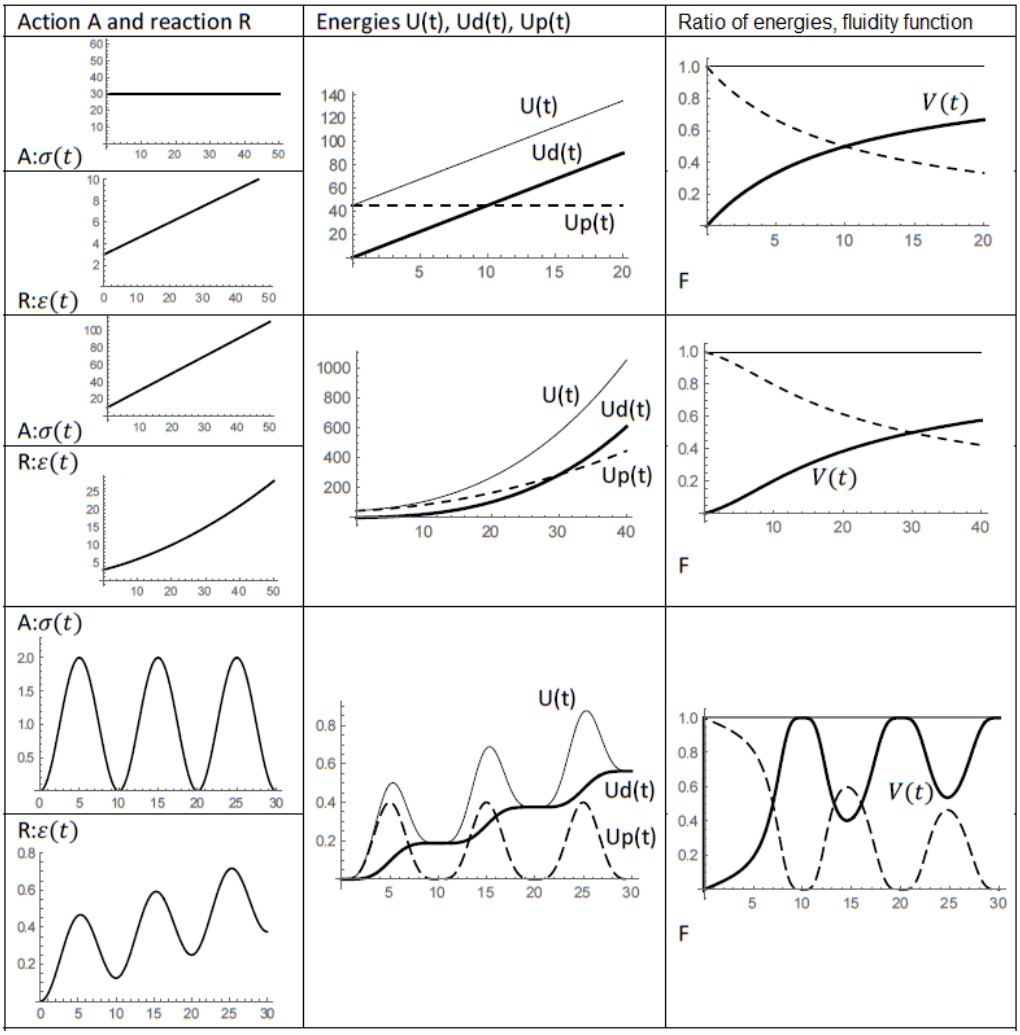

Figure 3.14: Maxwel viscoelastic model and its strain on stress material tests. Stress action and strain reaction functions - first column, deformation energy $U(t)$ and its split to stored (potential) energy $U_p(t)$ and dissipated energy $U_d(t)$ functions - 2nd column, fluidity function $V(t)$ - 3rd column.

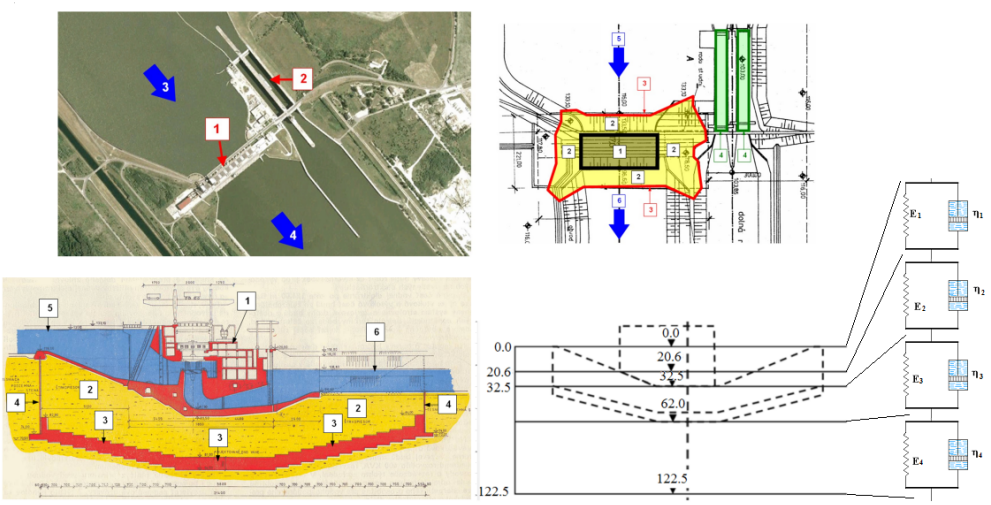

Figure 3.15: Hydropower station Gabčíkovo, top view and ground plan (upper images), cross-section, layered soil scheme and corresponding viscoelastic model - generalized Maxwell model (lower images), [15].

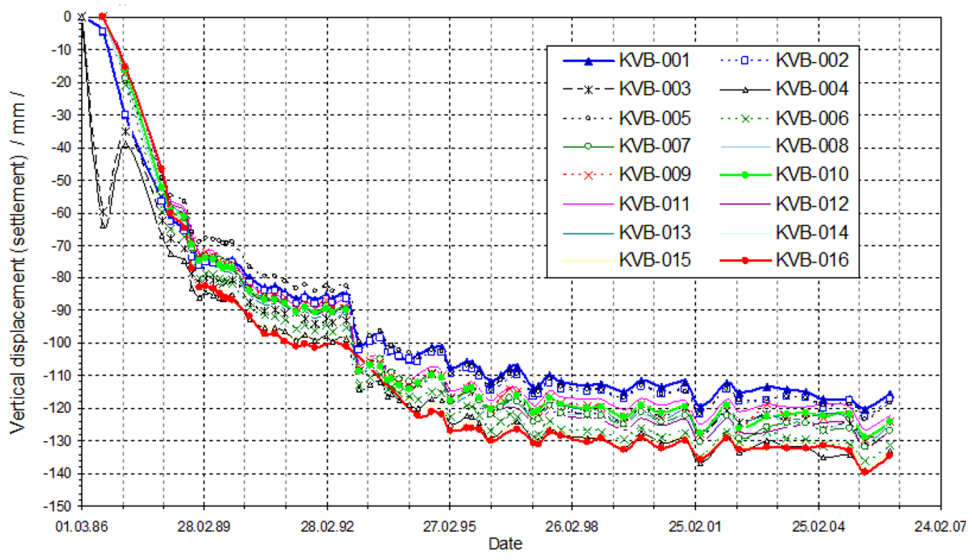

Figure 3.16: Secondary consolidation of the soil (creep settlement) beneath the Gabčíkovo hydropower station; in situ measurement values within twenty four years, [15].

and the viscoelastic model of the layered soil. The measurements of creep settlement within the so called *soil consolidation* is provided in Figure 3.16. The displacement starts from zero level down to the negative values, as can be seen in the ordinate range (zero to -150 mm). Hence, the strain trend (creep), called secondary consolidation of the soil, is a positive function increasing in time. Local oscillations of displacement values can be explained by the floating of the water level in the dam. More details about the soil consolidation can be found in [15]. The Kelvin chain model is chosen for the soil beneath the Gabčíkovo hydro-power station, see the viscoelastic model at the very right-hand side of Figure 3.15. Then, having the constitutive equation of the model, equation (3.114) for $n = 4$ and the material characteristics of all layers, we can start the theoretical investigation with the aim of predicting the soil behaviour under the future heavy load.

3.4.1 Chain viscoelastic models and Prony series

Chain viscoelastic models are created by n identical submodels linked *in parallel* or *in series*. Consequently, the order of the corresponding differential constitutive relation is determined by the configuration of the model. The number of irreducible viscous elements involved is decisive herein.
As it follows from the practice, a chain of several Maxwell models linked *in parallel* or a chain of several Kelvin-Voigt models linked *in series* can simulate properly the mechanical behaviour of many materials with sufficiently prescribed accuracy. As far as polymers are concerned, the preciseness of the model usually increases with every additional (M) member involved. But at the same time, the order of the differential governing equation increases as well, together with the complexity of computation and the number of necessary physical coefficients is higher too. These physical coefficients are usually stipulated by the inverse task based on the laboratory tests data. As performed in Figure 3.15, the soil with parallel homogeneous layers can be modelled by such a model. Chain uni-axial viscoelastic models

play a significant role in further investigation when enhanced to three dimensions.

Reducibility of viscous elements

In Figure 3.17 some reducible viscous elements are depicted. Due to their mutual parallel connection or connection in series, respectively, they can be reduced by the rules adequate to the parallel connection or the connection in series, respectively:

$$\begin{aligned} \eta &= \sum_{i=1}^{n} \eta_i, \\ \frac{1}{\eta} &= \sum_{i=1}^{n} \frac{1}{\eta_i}. \end{aligned} \tag{3.111}$$

Both reducing rules couple the particular physical characteris-

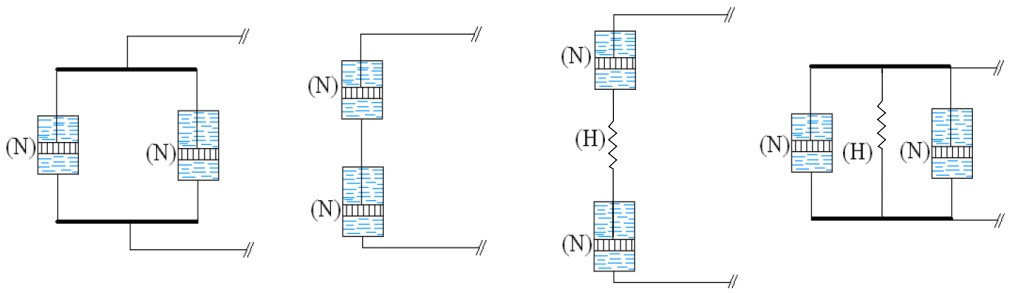

Figure 3.17: Reducible viscous elements - two (or several) viscous elements can be merged into one.

tics similarly to one value of the conditional stiffness (note the analogy in Chapter 3.2).

Generalized Maxwell model

The generalized Maxwell model (gM) is a complex model in which n Maxwell submodels are linked *in parallel*:
$(gM) = (M_1)|(M_2)|...|(M_i)|...|(M_n)$.
Eventually, regarding the matter behaviour, in order to ensure the reversion of the model to the initial stage, one additional Hookean element is usually joint in parallel to the generalized Maxwell model , [28, 33], see Figure 3.18)

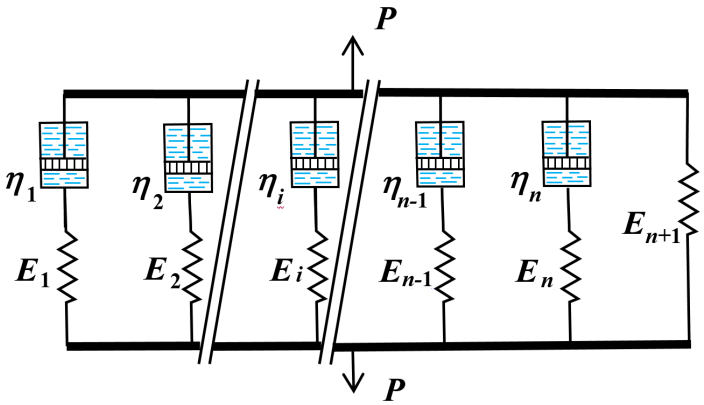

Figure 3.18: Generalized Maxwell model.

$(gM_H) = (M_1)|(M_2)|...|(M_i)|...|(M_n)|(H)$.
In accordance with considerations of the previous chapters we can set up the constitutive relation (3.25) by using the conditional stiffness tool dealt in Chapter 3.2. For the parallel connection of n Maxwell submodels, we have the plain sum $\hat{E} = \sum_{i=1}^{n} E_i'$ of conditional stiffness values E_i' of a particular i^{th} Maxwell submodel. By coupling this form with the individual stiffness of the Maxwell submodel $\frac{1}{E_i'} = \frac{1}{E_i} + \frac{1}{\eta_i D}$, we get the resulting *conditional stiffness* of (gM), [30]

$$\hat{E} = \sum_{i=1}^{n} \frac{E_i \eta_i D}{E_i + \eta_i D}, \tag{3.112}$$

and similarly the resulting *conditional stiffness* of (gM_H) is

$$\hat{E} = \sum_{i=1}^{n} \frac{E_i \eta_i D}{E_i + \eta_i D} + E_{n+1}. \tag{3.113}$$

Generalized Kelvin-Voigt model

Generalized Kelvin-Voigt model is a connection of n Kelvin-Voigt submodels in series, $(gK) = (K_1) - (K_2) - ... - (K_i) - ... - (K_n)$, see Figure 3.19. If we want to derive the constitutive relation of (gK)

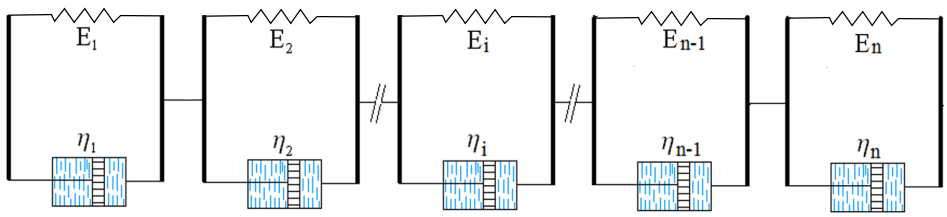

Figure 3.19: Generalized Kelvin-Voigt model.

conveniently, we just need to stipulate its conditional stiffness $\hat{E}$. From the serial connection of the particular (K) submodels, we have $\frac{1}{\hat{E}} = \sum_{i=1}^{n} \frac{1}{E_i'}$, with E_i' being the particular i^{th} member's conditional stiffness. The particular conditional stiffness of i^{th} Kelvin-Voigt submodel is $E_i' = E_i + \eta_i D$, compare with (3.24). So, the resulting *conditional stiffness of* (gK_n) is, [30]

$$\hat{E} = \frac{1}{\sum_{i=1}^{n} \frac{1}{E_i + \eta_i D}} \tag{3.114}$$

and its constitutive equation can be written down in the sense of the generalized Hook's law (3.25) and handled according the further need. Apparently,

$$\varepsilon(t) = \sum_{i=1}^{n} \frac{1}{E_i + \eta_i D} \sigma(t). \tag{3.115}$$

Prony series

The Prony series representation of (gM) is a tool utilized mainly within scientific society dealing with amorphous polymers. In this chapter, the Prony series are justified, and their relation with the constitutive equation of (gM) or (gM_H) is explained. We have the conditional stiffness (3.112) of (gM), thus we can first write down the constitutive relation in the sense of stress on strain explicit form $\sigma = \hat{E}\varepsilon$:

$$\sigma = \sum_{i=1}^{n} \frac{E_i \eta_i D}{E_i + \eta_i D} \varepsilon. \tag{3.116}$$

Designating the common denominator in (3.116), and multiplying the equation by it, we get the implicit (purely differential) constitutive relation of (gM):

$$\sigma \prod_{i=1}^{n}(E_i + \eta_i D) = \sum_{i=1}^{n} E_i \eta_i D \prod_{\substack{j=1 \\ j\neq i}}^{n}(E_j + \eta_j D)\varepsilon. \tag{3.117}$$

Afterwards, the inverted explicit (integro-differential) the strain on stress relation can be derived

$$\varepsilon = \frac{\prod_{i=1}^{n}(E_i + \eta_i D)}{\sum_{i=1}^{n} E_i \eta_i D \prod_{\substack{j=1 \\ j\neq i}}^{n}(E_j + \eta_j D)}\sigma. \tag{3.118}$$

For the sake of better clarity, and without loosing the generality at same time, we now proceed with the two-member (gM_2). We just put $n = 2$ in (3.112) getting the conditional stiffness of (gM_2) and we can easily write down even its implicit differential constitutive relation

$$\eta_1\eta_2\ddot{\sigma} + (E_1\eta_2 + E_2\eta_1)\dot{\sigma} + E_1E_2\sigma =$$

$$= \eta_1\eta_2(E_1 + E_2)\ddot{\varepsilon} + E_1E_2(\eta_1 + \eta_2)\dot{\varepsilon}.$$

Or, directly by using $\hat{E}$ or $\frac{1}{\hat{E}}$ we can either generally express the stress on strain or strain on stress explicitly. Let us recall that by indicating the particular action function (stress or strain respectively) course in time, we can compute the response (strain or stress respectively) directly from the resulting differential equation. Similarly as in the case of simple viscoelastic models, both the creep and the relaxation tests can be carried out, for the sake to obtain the individual rheological time-dependent material characteristics $E(t)$ and $C(t)$. Let us e.g. proceed with the relaxation test. We maintain the instantaneous strain $\varepsilon(t) = \varepsilon^*$ applied at the time instant $t_0 = 0$ at (gM_2), constant from that moment onwards. Initially, at the instant of imposing the strain, the stress jumps to

$$\sigma(0_+) = \left(E_1 + E_2\right)\varepsilon^* \tag{3.119}$$

and its initial stress rate is

$$\dot{\sigma}(0) = -\Big(\frac{E_1^2}{\eta_1} + \frac{E_2^2}{\eta_2}\Big)\varepsilon^*. \tag{3.120}$$

The linear homogeneous differential equation

$$a\ddot{\sigma} + b\dot{\sigma} + c\sigma = 0 \tag{3.121}$$

with constant coefficients $a = \eta_1\eta_2, b = (E_1\eta_2 + E_2\eta_1)$ and $c = E_1E_2$ corresponds to this relaxation test and together with initial condition (3.119) and (3.120) it gives the solution

$$\sigma(t) = \varepsilon^*\Big(E_1 e^{-\frac{E_1}{\eta_1}t} + E_2 e^{-\frac{E_2}{\eta_2}t}\Big) =$$

$$= \varepsilon^* \sum_{i=1}^{2} E_i e^{-\frac{E_i}{\eta_i}t} = \varepsilon^* \sum_{i=1}^{2} E_i e^{-\frac{t}{t_i}}. \tag{3.122}$$

where $t_i = -\frac{\eta_i}{E_i}$ is the i^{th} relaxation time. Enhanced to n and stipulated for (gM_H) it yields the so-called *Prony series* representation, [1, 5]

$$\sigma(t) = \Big(E_0 + \sum_{i=1}^{n} E_i e^{-\frac{t}{t_i}}\Big)\varepsilon^*. \tag{3.123}$$

Herein, the series $E_0 + \sum_{i=1}^{2} E_i e^{-\frac{t}{t_i}}$ stands as a time-dependent relaxation modulus of the model (see Chapter 3.2.2). With $t \to \infty$ the series (3.123) converges to $\sigma_0 = E_0\varepsilon^*$. In some discrete points, its values can be stipulated experimentally in the creep or relaxation test. These values are called the Prony coefficients, [35].

3.4.2 Recurrent models

A recurrent viscoelastic model is based on a repeating procedure of connecting the same members to the previous submodel. The recurrent (branch-chained) models can stand behind a great number of polymers, natural biomaterials and soils.

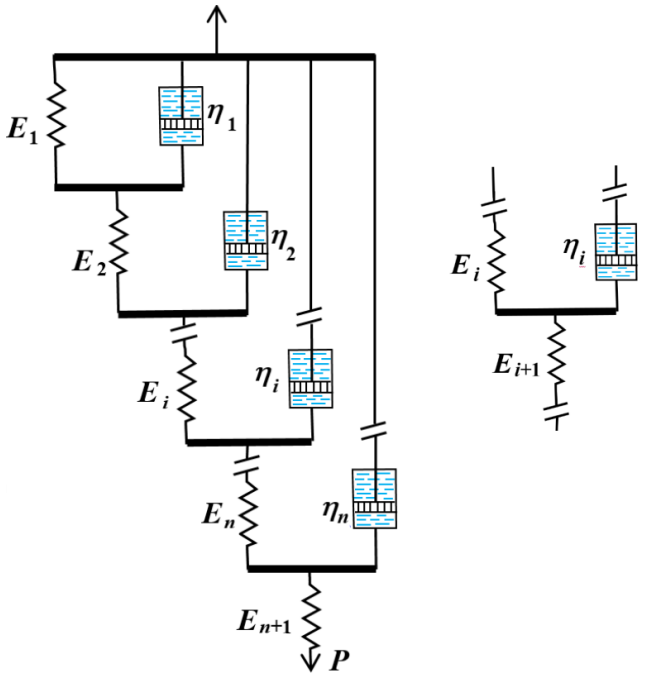

Figure 3.20: Recurrent (branch chain) Voigt model (*rV*) (left); characteristic recurrence unit (right).

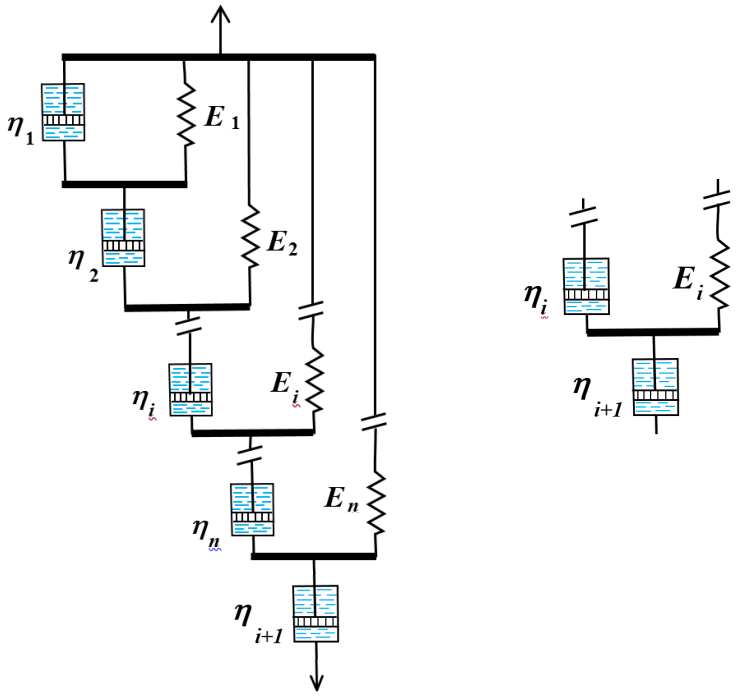

Figure 3.21: Recurrent (branch chain) Kelvin model (rK_{i+1}) (left); characteristic recurrence unit (right).

In this chapter we propose two models - the simplest recurrent configurations: In each step of the recurrence, one fundamental element is connected *in parallel* to the current model. This way, a new model arises. Then the other element is connected *in series* to the existing model. These two steps are repeated until the required level of recurrence is reached.
We can follow the creating procedure in Figure 3.20: at the beginning one viscous matter (N) is connected *in parallel* to the very first (H). The serial connection of another (H) follows afterwards, followed by next (N) connection *in parallel*, next (H) *in series*, etc., until a **recurrent Voigt model** $(rV_{i+1}) = [\ldots[(H_1)|(N_1)-(H_2)]|(N_2)-\cdots-(H_n)]|(N_n)-(H_{n+1})$ is built up of the required length.
Let us trace the conditional stiffness advancement of (rV_{i+1}) step by step: first, we have the conditional stiffness of a parallel connection of an i^{th} (N) matter with the previous submodel

of the i^{th} level with conditional stiffness $\bar{E}_i^V(D)$. After such connection the resulting conditional stiffness is

$$E_i'(D) = \bar{E}_i^V(D) + \eta_i D. \tag{3.124}$$

Afterwards, we realize the serial connection of (H_{i+1}) to the very recently arisen model. Using (3.69) yields

$$\frac{1}{\bar{E}_{i+1}^V(D)} = \frac{1}{E_i'(D)} + \frac{1}{E_{i+1}} = \frac{1}{\bar{E}_i^V(D) + \eta_i D} + \frac{1}{E_{i+1}} =$$

$$= \frac{\bar{E}_i^V(D) + \eta_i D + E_{i+1}}{\left[\bar{E}_i^V(D) + \eta_i D\right] E_{i+1}} \Rightarrow \bar{E}_{i+1}^V(D) = \frac{\left[\bar{E}_i^V(D) + \eta_i D\right] E_{i+1}}{\bar{E}_i^V(D) + \eta_i D + E_{i+1}}. \tag{3.125}$$

If we put $\bar{E}_1^V(D) = E_1$, we can affirm (3.125) as the *conditional stiffness recurrence form* of (rV_{i+1}).
If we start with (N) and alternate the parallel connection of (H) to the current model with the serial connection of (N) up to the $i+1^{st}$ level we get, see Figure 3.21, the **recurrent Kelvin model** of the length $i+1$: $(rK_{i+1}) =$
$=[\ldots[(N_1)|(H_1)-(N_2)]|(H_2)-\cdots-(N_i)]|(H_i)-(N_{i+1})$.
Again, the total conditional stiffness of i^{th} level is computed recurrently:

$$\frac{1}{\bar{E}_{i+1}^K(D)} = \frac{1}{E_i'(D)} + \frac{1}{\eta_{i+1} D} = \frac{1}{\bar{E}_i^K(D) + \eta_i D} + \frac{1}{\eta_{i+1} D} =$$

$$= \frac{\bar{E}_i^K(D) + \eta_i D + \eta_{i+1} D}{\left[\bar{E}_i^K(D) + \eta_i D\right] \eta_{i+1} D} \Rightarrow$$

$$\Rightarrow \bar{E}_{i+1}^K(D) = \frac{\left[\bar{E}_i^K(D) + \eta_i D\right] E_{i+1}}{\bar{E}_i^K(D) + \eta_i D + E_{i+1}}. \tag{3.126}$$

If we put $\bar{E}_1^K(D) = \eta_1 D$, we can affirm that (3.126) as the *conditional stiffness recurrence form* of (rK_{i+1}).
As it is undoubtedly apparent to the reader, many other recurrent models can be created by repeating any of steps, e.g. by

alternating the parallel and serial connections of submodels instead of the elementary models (*H*) and (*N*).

Algorithmization of deriving the constitutive equation

If we want to automatize the determining of the constitutive equation for some complex structured viscoelastic model, both the geometric and the physical relations have to be formalized and generalized. This was realized in the conditional stiffness concept. Let us summarize the required inputs and tools and yielded outputs that are reasonable within the algorithmization:

Inputs

- structural form of the model
- physical characteristics of all fundamental elements

Outputs

- pictogram of the model
- differential form of the constitutive relation

Tools

- possible reducibility of viscous elements disclosing and eliminating
- generalization - repetitive parts or recurrence detection, or splitting into blocks if no generalization exists
- conditional stiffness differential operators enfolding geometry feature, i.e. serial and parallel connection rules and particular physical equations of members involved

Upon the previous assumption, a "rheological calculator" was developed by [52], and a basic appearance revealing the functionality of this interactive program is provided in Figure 3.22.

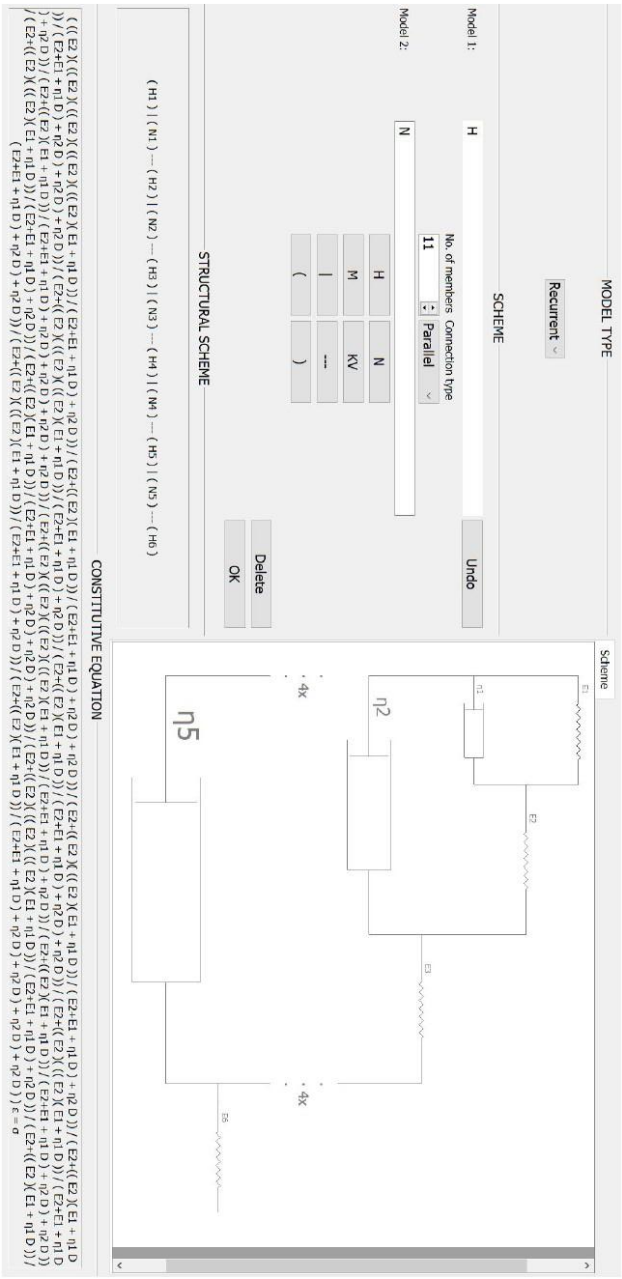

Figure 3.22: Calculator of constitutive equation of viscoelastic models, [52].

3.5 Three-dimensions. Anisotropy

In Chapter 3.1.3 we dealt with the global constitutive relation derivation of a viscoelastic model in the one-dimensional case. However, all considerations can be extended to two or three dimensions, where also an anisotropy can come into account.
From now on in this chapter, we will proceed with three dimensions, keeping the two-dimensional analogy in mind.
Generalized Hook's law in the sense of conditional stiffness in three dimensions for anisotropic matter can be written in the form

$$\boldsymbol{\sigma} = \hat{\mathbf{H}}\boldsymbol{\varepsilon}. \tag{3.127}$$

Herein, $\hat{\mathbf{H}}$, the elastic modulus tensor operator, includes both the volumetric and the deviatoric contribution to the deformation, as well as the configuration (geometry) of the entire model. It is apparent from Chapter 3.2, that, in general, it is a symmetric positive definite 4^{th}-order integro-differential tensor operator with symmetry in all directions, $\hat{\mathbf{H}} = H^{ijkl} = H^{klij} = H^{klji} = H^{lkji}$. It is worth recalling here Chapter 3.2.3, which performs how each constitutive relation, e.g. equation (3.127), can be rewritten in several mutually equivalent forms, employing a differential, an integral or an integro-differential tensor operator, [3].
Analogously to the one-dimensional case

$$\begin{aligned} \mathrm{K}^{(r)}\boldsymbol{\sigma} &= \mathbf{K}^{(s)}\boldsymbol{\varepsilon}, \\ \mathrm{Q}^{(r)}\boldsymbol{\varepsilon} &= \mathbf{Q}^{(s)}\boldsymbol{\sigma}, \end{aligned} \tag{3.128}$$

where

$$\begin{aligned} \mathrm{K}^{(r)} &= \textstyle\prod_{i=1}^{r}\left(\frac{\partial}{\partial t} + \kappa_i\right), \quad \mathrm{K}^{(0)} = 1, \\ \mathrm{Q}^{(r)} &= \textstyle\prod_{i=1}^{r}\left(\frac{\partial}{\partial t} + \lambda_i\right), \quad \mathrm{Q}^{(0)} = 1, \end{aligned} \tag{3.129}$$

are scalar differential operators and

$$\begin{aligned} \mathbf{K}^{(s)} &= \textstyle\sum_{i=1}^{s} \mathbf{K}^{(i)} \frac{\partial^i}{\partial t^i}, \\ \mathbf{Q}^{(r)} &= \textstyle\sum_{i=1}^{s} \mathbf{Q}^{(i)} \frac{\partial^i}{\partial t^i} \end{aligned} \tag{3.130}$$

are tensor differential operators (tensor of conditional stiffnesses) whose each component couples the time derivatives operators

with material properties, see the scalar interpretation e.g. in (3.24); κ_i and λ_i are the inverse values of the relaxation and the retardation time values, respectively. In the case when we have the tensor operator functions of relaxation $\mathbf{Y}(t)$ and retardation $\mathbf{J}(t)$ at our disposal, from previous creep and relaxation tests, we can express the constitutive relation (3.127) in the explicit stress on strain form or strain on stress form, integro-differential operators used:

$$\boldsymbol{\sigma}(t) = \int_0^t \mathbf{Y}(t-\tau)\frac{\partial \boldsymbol{\varepsilon}(t)}{\partial t} \tag{3.131}$$

or

$$\boldsymbol{\varepsilon}(t) = \int_0^t \mathbf{J}(t-\tau)\frac{\partial \boldsymbol{\sigma}(t)}{\partial t} \tag{3.132}$$

Let us recall that since the linear viscoelasticity theory is employed, the superposition principle is applied, and in the case of an isotropic material, the deviatoric and volumetric deformation can be treated separately. The analogy to the elastic deformation is used, where G is the deviatoric modulus, and K is the volumetric elastic modulus, [7, 8, 26]:

$$\begin{aligned} G &= \tfrac{E}{2(1+\mu)}, \\ K &= \tfrac{E}{3(1-2\mu)}. \end{aligned} \tag{3.133}$$

The time-dependent characteristics of the viscoelastic material can also be expressed in the form of the Prony series. For the generalized Maxwell model, we have

$$\begin{aligned} G(t) &= G_0^{\infty} + \textstyle\sum_{i=1}^{n} G_i e^{-\frac{t}{t_i}}, \\ K(t) &= K_0^{\infty} + \textstyle\sum_{i=1}^{m} K_i e^{-\frac{t}{t_i}}. \end{aligned} \tag{3.134}$$

Moreover, also the integral forms (3.131) and (3.132) can be split into the deviatoric and the volumetric part, [34]:

$$\int_0^t 2G(t-\tau)\frac{\mathrm{d}\boldsymbol{\varepsilon}_{dev}}{\mathrm{d}\tau}\mathrm{d}\tau + \mathbf{I}\int_0^t K(t-\tau)\frac{\mathrm{d}\boldsymbol{\varepsilon}_{vol}}{\mathrm{d}\tau}\mathrm{d}\tau. \tag{3.135}$$

3.6 Anisotropic material under pulsating load

Three-dimensional study

3.6.1 Pulsating load imposed on anisotropic viscoelastic body. Storage and loss moduli

Even nowadays, in accordance with the linear viscoelasticity theory, engineers usually consider a simplified isotropic homogeneous material. In such case, the relaxation tensor can be expressed, [43], as

$$H_{ijkl}(t) = \frac{1}{3}\Big[\mathcal{H}_2(t) - \mathcal{H}_1(t)\Big]\delta_{ij}\delta_{kl} + \frac{1}{2}\Big[\mathcal{H}_2(t)\Big]\Big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\Big), \quad (3.136)$$

which is the most general form of the isotropic 4^{th}-order tensor. Here, $\mathcal{H}_1(t)$ and $\mathcal{H}_2(t)$ are independent relaxation functions and δ_{ij} is the Kronecker operator

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Hence, the explicit constitutive relation σ - ε for an isotropic homogeneous material acquires the form

$$H_{ijkl}(t) = \int_{-\infty}^{t} \frac{1}{3}\Big[\mathcal{H}_2(t) - \mathcal{H}_1(t)\Big]\delta_{ij}\delta_{kl} +$$

$$+ \frac{1}{2}\Big[\mathcal{H}_2(t-\tau)\Big]\Big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\Big)\frac{\mathrm{d}\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau. \quad (3.137)$$

However, for more complex engineering problems it is often inevitable to deal with a physically more precise approach, that involves an anisotropy at all.
Analogously to the expression (3.47) in one-dimensional case, the three-dimensional explicit constitutive equation (Duhamel

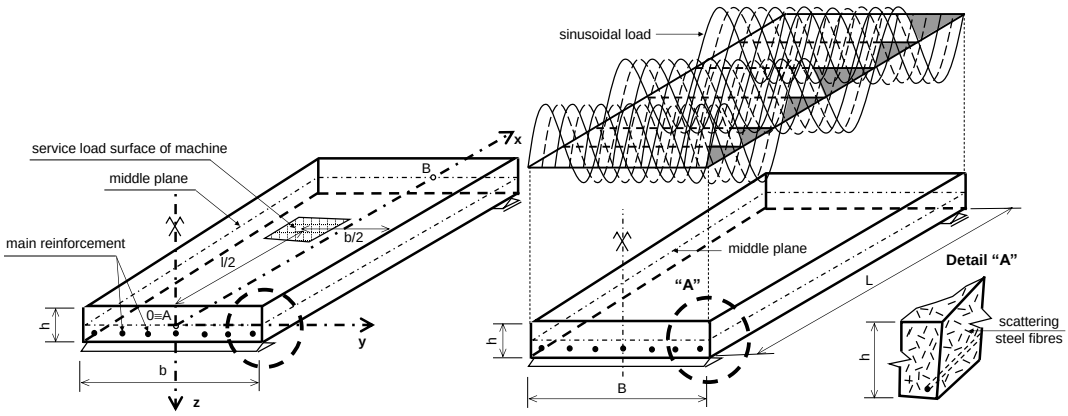

Figure 3.23: Scheme of analysed structural element, [46].

hereditary integral) can be derived, as well, where an anisotropic material is considered in general. Written componentwisely, it acquires the form, [3]:

$$\sigma_{ij}(t) = \int_{-\infty}^{t} H_{ijkl}\big(t-\tau\big)\frac{\mathrm{d}\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau, \tag{3.138}$$

where $\sigma_{ij}(t)$ and $\varepsilon_{kl}(t)$ are the 2^{nd}-order stress and strain tensors components, $H_{ijkl}(t)$ the components of the 4^{th}-order relaxation tensor acquiring zero value on the time interval $(-\infty, 0\rangle$. Assuming that the periodic sinusoidal strain load is an external action, we observe the reaction, the stress tensor function. Since the strain is a harmonic function of time, it can be expressed as

$$\varepsilon_{ij}(t) = \overset{\mathrm{o}}{\varepsilon}_{ij}\mathrm{e}^{\mathrm{i}\omega t}, \tag{3.139}$$

with imaginary unit $\mathrm{i} = \sqrt{-1}$, $\overset{\mathrm{o}}{\varepsilon}_{ij}$ being the strain amplitude and ω [rad/s] the angular frequency. Whenever needed, the angular frequency ω can be expressed in terms of the ordinary frequency ν [Hz]: $\omega = 2\pi\nu$.
Each component of the relaxation modulus tensor $H_{ijkl}(t)$ has a bounded variation on an arbitrary closed subinterval of $(-\infty, \infty)$ (see e.g. [7]). From the symmetry of both stress and strain tensors, the symmetry of the relaxation modulus follows in all

directions as well, [44]:

$$H_{ijkl}(t) = H_{jikl}(t) = H_{ijlk}(t). \tag{3.140}$$

From the thermodynamics we know that every process of a system under an unchanging load tends to its equilibrium. That is why it is worth to express the relaxation modulus, a function of time, as a sum of a tensor of non-negative constant values $\overset{\infty}{H}_{ijkl}$ - *equilibrium modulus* and a time-dependent *residual modulus* $\hat{H}_{ijkl}(t)$:

$$H_{ijkl}(t) = \overset{\infty}{H}_{ijkl} + \hat{H}_{ijkl}(t). \tag{3.141}$$

As the process of the system tends towards the equilibrium stage the residual modulus

$$\hat{H}_{ijkl}(t) \to 0 \quad \text{for} \quad t \to \infty. \tag{3.142}$$

and only the time independent tensor $\overset{\infty}{H}_{ijkl}$ remains in (3.141), standing as the material modulus of the stationary state constitutive relation. Each component of the equilibrium modulus tensor is always positive for solid bodies and equal to zero for liquids, e.g. [34].
When substituting (3.139) and (3.141) to (3.138), we have the explicit constitutive equation for a pulsating load in the form

$$\sigma_{ij}(t) = \int_{-\infty}^{t} \Big(\overset{\infty}{H}_{ijkl} + \hat{H}_{ijkl}(t-\tau)\Big) \mathrm{d} \frac{\overset{\circ}{\varepsilon}_{kl} e^{\mathrm{i}\omega\tau}}{\mathrm{d}\tau} \mathrm{d}\tau. \tag{3.143}$$

The first addend $\overset{\infty}{H}_{ijkl}$ in the integral (3.143) can be computed in sense of Stieltjes yielding a function of ω whilst in the second addend the derivative of $\overset{\circ}{\varepsilon}_{kl} e^{\mathrm{i}\omega\tau}$ with respect to time can be evaluated τ:

$$\sigma_{ij}(t) = \overset{\infty}{H}_{ijkl} \overset{\circ}{\varepsilon}_{ij} e^{\mathrm{i}\omega t} + \mathrm{i}\omega \overset{\circ}{\varepsilon}_{ij} \int_{-\infty}^{t} \hat{H}_{ijkl}(t-\tau) e^{\mathrm{i}\omega\tau} \mathrm{d}\tau. \tag{3.144}$$

Let us proceed with (3.144) by using the substitution $\chi = t - \tau$. We obtain

$$\sigma_{ij}(t) = \overset{\circ}{\varepsilon}_{ij} e^{\mathrm{i}\omega t} \Big[\overset{\infty}{H}_{ijkl} - i\omega \int_{0}^{\infty} \hat{H}_{ijkl}(\chi) e^{\mathrm{i}\omega\chi} \mathrm{d}\chi \Big].$$

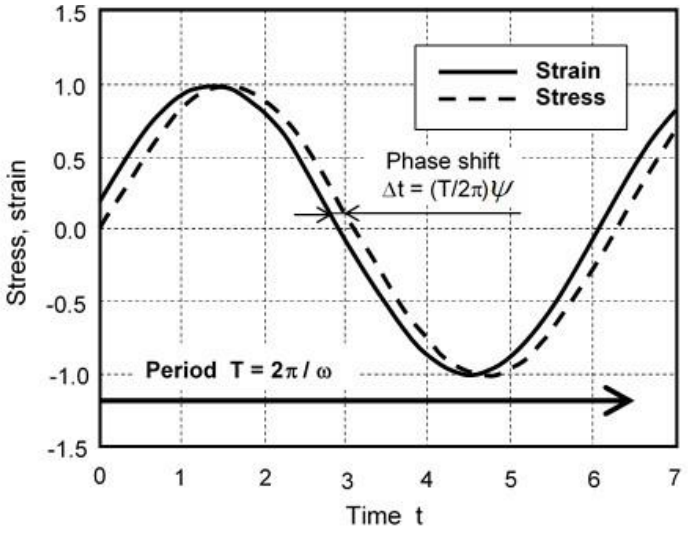

Figure 3.24: Stress and strain in time t with the phase shift; T is the period of action and reaction, ω [rad/s] is the angular frequency, ψ [rad] is the phase angle between the stress and strain sinusoid, [39].

Now, when employing the Euler form

$$e^{-\mathrm{i}\omega\tau} = \cos(\omega\tau) - \mathrm{i}\sin(\omega\tau), \tag{3.145}$$

we get

$$\sigma_{ij}(t) = \overset{\circ}{\varepsilon}_{ij} e^{\mathrm{i}\omega t}\Big[\overset{\infty}{H}_{ijkl} + \omega\int_0^\infty \hat{H}_{ijkl}(\chi)\sin(\omega\chi)\mathrm{d}\chi + \\ + \mathrm{i}\omega\int_0^\infty \hat{H}_{ijkl}(\chi)\cos(\omega\chi)\mathrm{d}\chi\Big]. \tag{3.146}$$

Physically speaking, if we do not consider damping at the mechanical reaction of the model, then the stress response corresponding to a certain stationary pulsating deformation load acquires the same periodical character as the strain action. Hence, the amplitude stays at the same value in time too, [37]. Indeed, the response delay after the load, expressed as the time shift

$$\Delta t = T\frac{\alpha}{2\pi}$$

appears, with $T = \frac{2\pi}{\omega}$ being the period and α the phase angle (see Figures 3.24 and 3.25). It means that for this special kind of load, the general stress-strain relation (3.138) takes the form

$$\sigma_{ij}(t) = H^*_{ijkl}(\omega)\overset{\circ}{\varepsilon}_{ij}e^{\mathrm{i}\omega t}. \tag{3.147}$$

Herein, the expression in square brackets in (3.146)

$$\overset{\infty}{H}_{ijkl} + \omega\int_0^\infty \hat{H}_{ijkl}(\chi)\sin(\omega\chi)\mathrm{d}\chi + \mathrm{i}\omega\int_0^\infty \hat{H}_{ijkl}(\chi)\cos(\omega\chi)\mathrm{d}\chi$$

is known as the *dynamical modulus* and is denoted by $H^*_{ijkl}(\omega)$ within the viscoelasticity theory. Since σ_{ij} is a complex function of frequency ω, it is natural to split it into the real and the imaginary part

$$H^*_{ijkl}(\omega) = \bar{H}_{ijkl}(\omega) + \mathrm{i}\bar{\bar{H}}_{ijkl}(\omega). \tag{3.148}$$

Moreover, the real and the imaginary part of the dynamical modulus have their names - *storage modulus* and *loss modulus*, respectively. Both moduli are reasoned physically too, [22, 23]. Accordingly, it follows from (3.146) that in our special load type, the pulsating load, the storage and the loss moduli are

$$\bar{H}_{ijkl}(\omega) = \overset{\infty}{H}_{ijkl} + \omega\int_0^\infty \hat{H}_{ijkl}(\chi)\sin(\omega\chi)\mathrm{d}\chi, \tag{3.149}$$

$$\bar{\bar{H}}_{ijkl}(\omega) = \omega\int_0^\infty \hat{H}_{ijkl}(\chi)\cos(\omega\chi)\mathrm{d}\chi \tag{3.150}$$

respectively. Moreover, the ratio

$$\frac{\bar{\bar{H}}(\omega)}{\bar{H}(\omega)} = \tan\alpha(\omega) \tag{3.151}$$

represents the so-called *loss tangent*, [46].

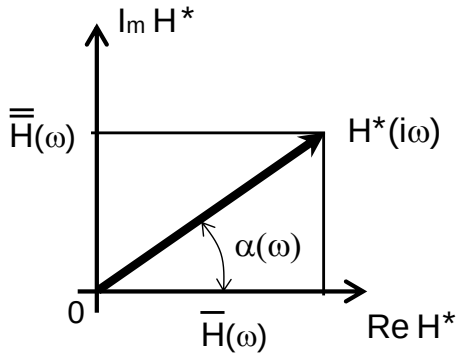

Figure 3.25: Loss tangent.

Remark
In [7], there is a complaint of some misunderstandings in the terminology regarding the term dynamical modulus $H^*_{ijkl}(\omega)$.
Namely, the term "dynamical" does not fit here well, since it does not indicate explicitly whether or not inertial members are retained in the momentum equations. Maybe the name "undamped pulsation modulus" would be more appropriate in this case as suggested by [13].
For physical purposes, we are supposed to examine the impact of possible limiting values of action frequency on the resulting dynamical modulus. For this purpose, recalling (3.142) we rearrange the integrals in (3.149) and (3.150) for a moment by integrating by parts getting

$$\bar{H}_{ijkl}(\omega) = \overset{\infty}{H}_{ijkl} + \hat{H}_{ijkl}(0) + \int_0^\infty \frac{\mathrm{d}\hat{H}_{ijkl}(\chi)}{\mathrm{d}\chi}\cos(\omega\chi)\mathrm{d}\chi, \qquad (3.152)$$

and

$$\bar{\bar{H}}_{ijkl}(\omega) = -\int_0^\infty \frac{\mathrm{d}\hat{H}_{ijkl}(\chi)}{\mathrm{d}\chi}\sin(\omega\chi)\mathrm{d}\chi. \qquad (3.153)$$

After such a rearrangement, we can conveniently follow the impact of limiting frequency values:

- For $\omega \to 0$ we have the static load

$$\bar{H}_{ijkl}(0) = \overset{\infty}{H}_{ijkl} = H_{ijkl}(t)|_{t\to\infty} \qquad (3.154)$$

$$\bar{\bar{H}}_{ijkl}(0) = 0. \tag{3.155}$$

- For a very high frequency $\omega \to \infty$ we get the impact load. After substitution $\omega\chi = \tau$ we get

$$\lim_{b\to\infty} \bar{H}_{ijkl}(b) = \overset{\infty}{H}_{ijkl} + \hat{H}_{ijkl}(0) = H_{ijkl}(t)|_{t\to 0} \tag{3.156}$$

$$\lim_{\omega\to\infty} \bar{\bar{H}}_{ijkl}(\omega) = 0. \tag{3.157}$$

The formulas (3.154) and (3.157) represent the limit values of the real and imaginary parts of the complex relaxation modulus. Since the zero frequency corresponds to a static load, it is apparent that the imaginary part (3.154) has to be zero. Relations (3.156) and (3.157) document the fact that in the case of a very high frequency of excitation, the imaginary part of the complex module converges to zero. As far as the mechanical response is concerned, the system behaves as an elastic anisotropic body.
It is worth emphasizing that forms (3.149) and (3.150) physically perform the dependence between the frequency and the mechanical properties of the body expressed in the form of the relaxation moduli. Therein, tensor $\bar{H}_{ijkl}(\omega)$ is a component of the stress-strain ratio in the direction of the deformation phase, [42], whereas the tensor $\bar{\bar{H}}_{ijkl}(\omega)$ is declined from that direction by the angle of 90°.
Sometimes there is a need to express the constitutive equation (3.147) with the phase shift φ directly incorporated, [22]:

$$\sigma_{ij}(t) = H^{*}_{ijkl}(\omega)\overset{\circ}{\varepsilon}_{kl} e^{\mathrm{i}(\omega t+\varphi)}.$$

In such a case, it is explicitly given that compared to the periodical deformation load, the stress response is delayed by φ (see Figure 3.26). The delay can be computed, measured and recorded. Nevertheless, we are mostly focused on the mechanical behaviour of a device in terms of the danger of its damage. Thus there is no special need to trace the time shift between the action and the reaction. The relations (3.154) - (3.157) document the fact that in both zero and infinity limiting cases of frequency, the behaviour of an anisotropic body is almost purely elastic.

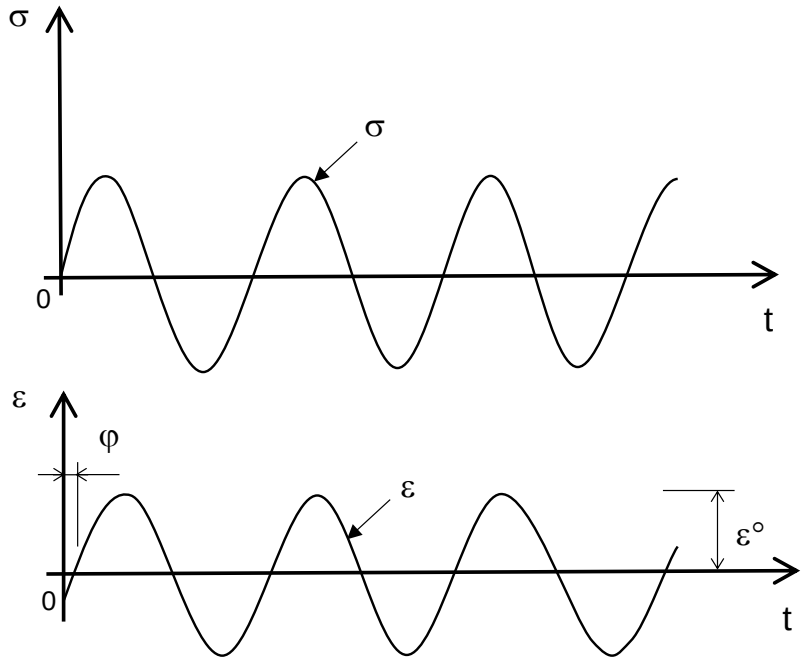

Figure 3.26: Periodical course of stress σ and strain ε of a viscoelastic material, [42].

3.6.2 Viscoelastic body under a pulsating load. Fourier transform.

Integral transform

Integral transforms are used for solving various science and technology tasks. The Fourier integral transform allows an aperiodic function to be expressed as an integral sum over a continuous range of frequencies, [6]. It is commonly used in linear tasks of the continuum mechanics. The form of direct Fourier transform is, see e.g. [22]:

$$\mathcal{F}(f(t)) = \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-\mathrm{i}\omega t} \mathrm{d}t. \tag{3.158}$$

An *original* - the function $f(x)$ is transformed to its *image*, $\tilde{f}(x)$. The inverse Fourier transform reverts the function transformed by the Fourier transform (image) to the original form:

$$\mathcal{F}^{-1}(\tilde{f}(\omega)) = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{\mathrm{i}\omega t}\mathrm{d}\omega. \tag{3.159}$$

Fourier transform is often used in the form of its cosine and sine forms as well. Apparently, both the sine and the cosine Fourier transform is yielded by a split of Fourier transform (3.158) by using the Euler form (3.145); and alike the inverse Fourier transform (3.159):

$$\mathcal{F}_c\Big(f(t)\Big) = \tilde{f}(\omega) = \int_0^{\infty} f(t)\cos(\omega t)\mathrm{d}t \tag{3.160}$$

$$\mathcal{F}_c^{-1}\Big(\tilde{f}(\omega)\Big) = f(t) = \frac{2}{\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)\cos(\omega t)\mathrm{d}\omega \tag{3.161}$$

$$\mathcal{F}_s\Big(f(t)\Big) = \tilde{f}(\omega) = \int_0^{\infty} f(t)\sin(\omega t)\mathrm{d}t \tag{3.162}$$

$$\mathcal{F}_s^{-1}\Big(\tilde{f}(\omega)\Big) = f(t) = \frac{2}{\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)\sin(\omega t)\mathrm{d}\omega \tag{3.163}$$

In our investigation we focus on the mechanical response of a viscoelastic pad exposed to a pulsating load e.g. due to the rotating machine put on it. Regarding the spinning, the frequency is more distinctive factor than time itself as far as the mechanical response is concerned. Obviously, we use those sine and cosine Fourier transforms as a useful tool for building up a switch between the constitutive relations written in the sense of time and those written in the sense of frequency.
The storage and loss moduli (3.149) and (3.150) can be regarded as Fourier images, hence the originals can be derived by using (3.163) and (3.161):

$$\hat{H}_{ijkl}(t) = \frac{2}{\pi}\int_0^{\infty}\frac{\bar{H}_{ijkl}(\omega) - \overset{\infty}{H}_{ijkl}}{\omega}\sin(\omega t)\mathrm{d}\omega, \tag{3.164}$$

$$\hat{H}_{ijkl}(t) = \frac{2}{\pi}\int_0^{\infty}\frac{\bar{\bar{H}}_{ijkl}(\omega)}{\omega}\cos(\omega t)\mathrm{d}\omega. \tag{3.165}$$

Next, for $t > 0$, we can rearrange (3.164) as follows:

$$\hat{H}_{ijkl}(t) = \frac{2}{\pi}\int_0^{\infty} \frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin(\omega t)\mathrm{d}\omega - \frac{2}{\pi}\overset{\infty}{H}_{ijkl}\int_0^{\infty}\frac{\sin(\omega t)}{\omega}\mathrm{d}\omega =$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin(\omega t)\mathrm{d}\omega - \overset{\infty}{H}_{ijkl},$$

where we have taken that $\int_0^{\infty}\frac{\sin(\omega t)}{\omega}\mathrm{d}\omega = \frac{\pi}{2}$ into account. Now, we have

$$\hat{H}_{ijkl}(t) + \overset{\infty}{H}_{ijkl} = H_{ijkl}(t) = \frac{2}{\pi}\int_0^{\infty}\frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin(\omega t)\mathrm{d}\omega, \qquad (3.166)$$

the ready-to-use relaxation coefficient. Substituting it in the stress-strain relation (3.138) yields the resulting explicit constitutive equation for an anisotropic material subjected to periodic pulsating strain load

$$\sigma_{ij}(t) = \frac{2}{\pi}\int_{-\infty}^{t}\int_0^{\infty}\frac{\bar{H}_{ijkl}(\omega)}{\omega}\sin\Big(\omega(t-\tau)\Big)\mathrm{d}\frac{\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\omega\mathrm{d}\tau. \qquad (3.167)$$

Consequently, by using the Fourier transform we get the Fourier image for the stress tensor function $\tilde{\sigma}_{ij}(\omega)$

$$\tilde{\sigma}_{ij}(\omega) = \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{t} H_{ijkl}(t-\tau)\frac{\mathrm{d}\varepsilon_{kl}(\tau)}{\mathrm{d}\tau}\mathrm{d}\tau\Big]e^{-i\omega t}\mathrm{d}t. \qquad (3.168)$$

Well, whenever having $\tilde{\sigma}_{ijkl}(t)$, by using the inverse Fourier transform we get the Fourier original $\sigma_{ij}(\omega)$. Moreover, when we couple the transforms (3.159) with (3.148) - (3.150), we can express the constitutive equation (3.168) in the shortened form

$$\tilde{\sigma}_{ij}(\omega) = H^{*}_{ijkl}(\omega)\tilde{\varepsilon}_{kl}(\omega). \qquad (3.169)$$

Considering the type of solved problems, we can now use constitutive relations either in the sense of time t (3.147) or in the sense of frequency ω (3.169).
The elements of the relaxation spectrum set of an anisotropic

material, are vectors with their components corresponding to the particular directions. In our case, the case of orthotropy (see Figure 3.23), the two-dimensional vectors are present as elements of both the retardation and relaxation spectra. The corresponding components can be measured in a lab, each one separately, by imposing the load in the corresponding direction. The resulting more "solid-like" behaviour is performed alongside the main reinforcement (x-axis in the Figure 3.23) and more "liquid-like" behaviour in all perpendicular directions.

Chapter 4

VISCO ELASTO PLASTICITY

4.1 Plasticity in rheological models. Variation inequality

In Chapter 2.1.2, a plastic element was introduced among the basic rheological matters. Since its mechanical behaviour exhibits singularity, the constitutive relation cannot be expressed by a function; therefore the Saint-Venant variational inequality is exploited, see the third form in Definition 3 in Chapter 2.1.2. When a plastic element is connected *in parallel* with an elastic one, the geometrical equations of $(H)|(StV)$ arise:

$$\begin{aligned} \varepsilon &= \varepsilon_H = \varepsilon_{StV} \\ \sigma &= \sigma_H + \sigma_{StV}. \end{aligned} \tag{4.1}$$

Geometric equation combined with the attached physical relations, give a variational inequality, [20, 19]:

$$\begin{aligned} &\sigma_H = E\,\varepsilon_H \\ &\dot{\varepsilon}_{StV}\big(\sigma_{StV} - \tilde{\sigma}\big) \geq 0, \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle. \end{aligned} \tag{4.2}$$

can be combined into one as follows:

$$\dot{\varepsilon}\,(\sigma - E\varepsilon - \tilde{\sigma}) \geq 0, \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle. \tag{4.3}$$

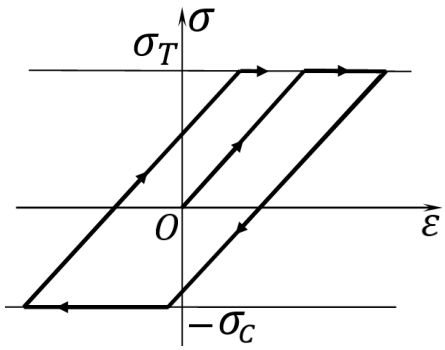

Figure 4.1: Stop operator.

And besides - since the only contribution to the entire potential energy comes from the elastic element, the potential energy is $U_p = \frac{1}{2}E\varepsilon_H^2$.
The thermodynamic consistency of the model follows actually from the variational inequality (4.3) (with $\tilde{\sigma} = 0$).
For $(H)-(StV)$, the combination *in series* of both (H) and (StV), we have the corresponding geometric equations

$$\begin{aligned} \varepsilon &= \varepsilon_H + \varepsilon_{StV}, \\ \sigma &= \sigma_H = \sigma_{StV}. \end{aligned} \tag{4.4}$$

Coupling the geometric equations with the physical relations of particular involved members (2.7) and (4.3), we can easy derive the resulting global governing variational inequality, which is the constitutive equation of the model $(H)-(StV)$:

$$\left(\dot{\varepsilon} - \frac{1}{E}\dot{\sigma}\right)(\sigma - \tilde{\sigma}) \geq 0, \quad \forall \tilde{\sigma} \in \langle -\sigma_C, \sigma_T \rangle. \tag{4.5}$$

The potential energy is $U_p = \frac{1}{2}E\varepsilon_H^2$ again and, the thermodynamic consistency follows directly from (4.5) when $\tilde{\sigma} = 0$ is taken. The yielded variational inequalities in the cases, i.e. parallel and serial connection of plastic matter, are of the same type and according to the Theorem 1.9 in [20], there exists a unique solution to each of the variational inequalities, which are expressed by using the *Play operator* or the *Stop operator*

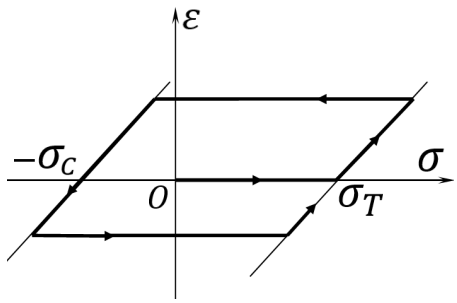

Figure 4.2: Play operator.

respectively; both hysteresis operators are depicted in Figures 4.1 and 4.2.
Hysteresis operators are rate-independent. They involve the memory effects, which means that the current stage of the system depends on its previous history.
More details about hysteresis operators can be found e.g. in [20, 49, 4, 18].

4.1.1 Viscoelastoplastic models and applications

Within the rheology of composite materials, there exists a special group with materials based on silicates. This group involves all kinds of concretes and reinforced concretes.
Due to the load subjected (mechanical, thermo-mechanical and chemo-mechanical), on the constructions built from concrete or steel-reinforced concrete, first an instantaneous deformation occurs, then short-term (maturing) and long-term deformations continue. The long-term change persists even within several years. When the fresh standard concrete mixture is poured into a mould, it immediately begins to solidify along with chemical processes, resulting in a volume decrease independent of the load. Additionally, the imposed load activates a creep, and consequently the hysteresis in the response (dependent on the load/unload period) is always exhibited. The creep of concrete is of high significance among civil engineers, and it should be

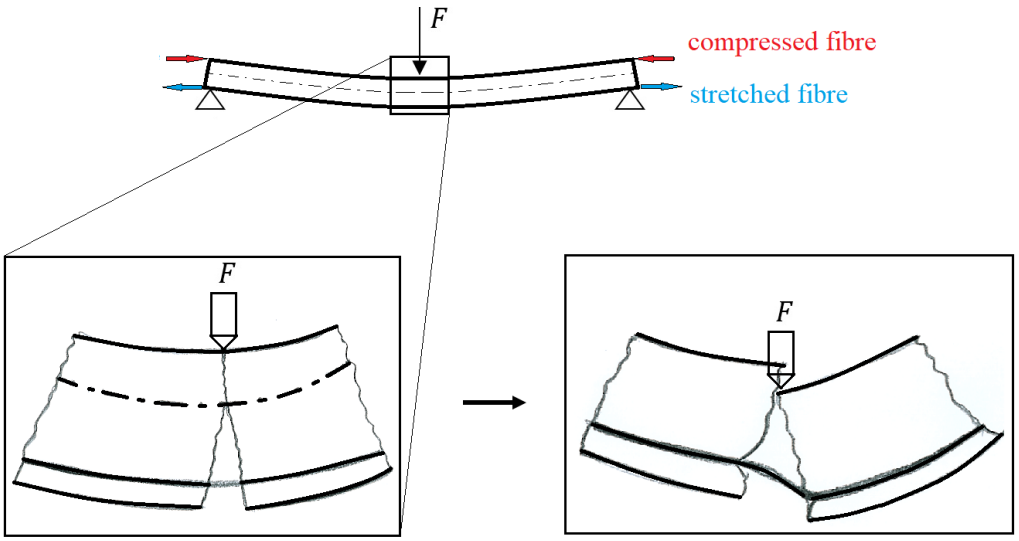

Figure 4.3: Transversally overloaded concrete beam; relocation of the compressed and stretched fibres domain (left), rupture due to the threshold achieved, [19].

predicted properly.

4.1.2 Concrete mechanical response to heavy load

It is a well-known fact that the compression strength of concrete is much higher than its tensile strength. That is also the reason, why the reinforcement is always placed not in the middle, but on the side where the tension of the material fibres is expected. Beams are reinforced at the bottom, balcony consoles at the top, etc.
The frequent deformation of the mature concrete occurs not only as a consequence of the load but also due to changes in temperature or humidity of the ambient. Moreover, since the concrete is not homogeneous, as its components are tiny stones, the deformations are often accompanied by microcracks due to the stress peaks inside of it. The volumetric deformation of concrete is not negligible - when exposed to compression, the

volume of concrete decreases, and when exposed to tension, the volume increases. When the load magnitude oscillates, e.g. tension and compression alternate, hysteresis occurs, [42].
Nevertheless, for normal operating regime, the linear behaviour of concrete is phenomenologically ascertained, and within the computations a proportionality of the mechanical load and the response of concrete is assumed; the superposition principle is held.

4.1.3 Simple rheological model of concrete with failure. Derivation of the constitutive relation

The simplest viscoelastoplastic model of specific concrete can stand behind the overloaded concrete as far as the mechanical behaviour is concerned, see Figure 4.4. Since a plastic change is expected (see Figure 4.3), the elastic element involved in the model representing the linear bahaviour, is amended with a serial connection of the assemblage of one viscous and one rigid-plastic elements merged *in parallel*, [42]. Accordingly, the rheological model's structural form is $(concrete) = (H) - [(N)|(StV)]$, [42].
The basic physical relations related to the particular elementary members are taken as in Section 2.1.2. The needed parameters are: E - Young elastic modulus of (H), η - the viscous coefficient of (N), σ_T and σ_C - the stress tensile and compressive thresholds of the stress of (StV). The derivation of the constitutive relation begins with assembling of the corresponding geometric and physical equations, and the subindexes of the strain and the stress variables are used to indicate the incidence with the particular member. The geometrical equations of the concrete, following from its configuration are:

$$\begin{aligned} &\varepsilon = \varepsilon_H + \varepsilon_N \\ &\varepsilon_N = \varepsilon_{StV} \\ &\sigma = \sigma_H = \sigma_N + \sigma_{StV}, \end{aligned} \tag{4.6}$$

where $\sigma_{StV} \in \langle -\sigma_C, \sigma_T \rangle$. The physical equations are (4.2) and (2.11). As far as the energy is concerned, even here we see

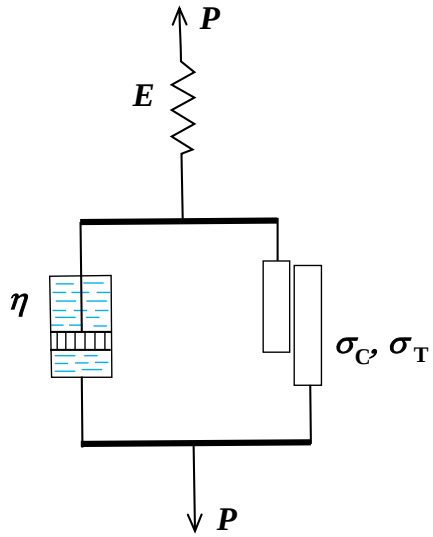

Figure 4.4: Simplified rheological model of the concrete, $(concrete) = (H) - \big[(N)|(StV)\big]$.

that the elastic energy $U_p = \frac{1}{2} E \varepsilon_H^2$ is the only energy of the entire system that is kept for future recovery. The model is thermodynamically consistent. We get the thermodynamic consistency

$$\dot{\varepsilon}\sigma - \frac{1}{2} E \dot{\varepsilon}_H^2 \geq 0$$

from the variational inequality (4.10) for $\check{\sigma} = 0$.
Deriving the overall constitutive relation of the model means establishing the relation between global stress and global strain of the model expressed only in the sense of the physical parameters of the particular elements. We are supposed to couple the geometric and physical relations to eliminate all sub-indexed variables:

$$\begin{aligned} &\sigma_H = E\varepsilon_H, \\ &\sigma_N = \eta\, \dot{\varepsilon}_N \implies \varepsilon_N = \varepsilon - \varepsilon_H = \varepsilon - \tfrac{1}{E}\sigma_H, \\ &\sigma = E\varepsilon_H = \eta\, \dot{\varepsilon}_N + \sigma_{StV}, \\ &\sigma_{StV} = \sigma - \eta\, \dot{\varepsilon_N} = \sigma - \eta\dot{\varepsilon} + \tfrac{\eta}{E}\, \dot{\sigma}. \end{aligned} \tag{4.7}$$

The Saint-Venant variational inequality of the rigid-plastic matter indexed by (StV) within the rheological model of concrete

is

$$\dot{\varepsilon}_{StV}\,(\sigma_{StV}-\tilde{\sigma})\geq 0,\forall\tilde{\sigma}\in\langle-\sigma_C,\sigma_T\rangle. \tag{4.8}$$

And having in mind that

$$\varepsilon_{StV}=\varepsilon_N=\varepsilon-\varepsilon_H=\varepsilon-\frac{\sigma}{E}, \tag{4.9}$$

we get the resulting variational inequality

$$\left(\dot{\varepsilon}-\frac{\dot{\sigma}}{E}\right)\left(\sigma-\eta\dot{\varepsilon}+\frac{\eta}{E}\dot{\sigma}-\tilde{\sigma}\right)\geq 0\quad\forall\tilde{\sigma}\in\langle-\sigma_C,\sigma_T\rangle \tag{4.10}$$

With the notation

$$\dot{v}=\dot{\varepsilon}-\frac{\dot{\sigma}}{E}, \tag{4.11}$$

we can rewrite (4.10) as

$$\dot{v}\big(\sigma-\eta\dot{v}-\tilde{\sigma}\big)\geq 0,\quad\forall\tilde{\sigma}\in\langle-\sigma_C,\sigma_T\rangle. \tag{4.12}$$

If the expression $\sigma-\eta\dot{v}$ in (4.12) belongs to the open interval $(-\sigma_C,\sigma_T)$, the expression in the first brackets can acquire either positive or negative values when $\tilde{\sigma}$ changes. That is why $\dot{v}=0$ has to be fulfilled in this case.
Moreover, we have to inspect both tensile and compressive margins as well:

- When the compressive margin $\sigma-\eta\dot{v}=-\sigma_C$ is taken, it is evident that for all $\tilde{\sigma}\in\langle-\sigma_C,\sigma_T\rangle:\sigma-\eta\dot{v}-\tilde{\sigma}\leq 0$. Therefore, to fulfil the inequality (4.12), $\dot{v}\leq 0$ has to be valid. Hence,

$$\dot{\varepsilon}-\frac{\dot{\sigma}}{E}\leq 0. \tag{4.13}$$

- Similarly, if $\sigma-\eta\dot{v}=-\sigma_T$ then it has to hold $\dot{v}\geq 0$, or

$$\dot{\varepsilon}-\frac{\dot{\sigma}}{E}\geq 0. \tag{4.14}$$

Indeed, by using the play operator, [20], both marginal cases can be coupled in one relation

$$\dot{v}=\frac{1}{\eta}(\sigma-\mathcal{P}(\sigma)), \tag{4.15}$$

written in terms of ν, where $\mathcal{P}$ is the projection on the interval $\langle -\sigma_C, \sigma_T \rangle$ in the sense of the convex analysis, [20]. Finally, when we carry out the backward substitution of ν given by (4.11) we get the relation $\sigma \sim \varepsilon$, the required global constitutive relation

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{1}{\eta}(\sigma - \mathcal{P}(\sigma)). \tag{4.16}$$

representing the mechanical behaviour of the model.

Conclusion

Nowadays, when new and new materials of certain specific properties are required by medicine, tissue engineering, technologies, structural and civil engineering, the food industry, etc., material science is experiencing its boom together with related disciplines of chemistry. Rheology plays a significant role here. Each year, plenty new materials are invented by engineers. A lot of them end up being of no use, some of them are excellent. All eligible materials have to be tested properly, as far as their mechanical behaviour is concerned, prior to their placement in service. The type of expected load, as well as other assumed non negligible physical or chemical circumstances, have to be taken into account carefully. And this is the point where rheology reveals its importance. As soon as the constitutive relation of the material is known, either the relaxation modulus $E(t)$ or the creep compliance $C(t)$ is employed, and various tests can be executed theoretically. This way, the behaviour prediction is provided, and the life span of the material can be approximately estimated together with its possible overloading either due to too high magnitude of the load or due to the time length of the load (though of low magnitude) exposition. Of course, the accordance with appropriate laboratory tests is required as well. Underestimation of both theoretical and laboratory investigation can bring fatal consequences in failures of materials. Let us just think of biocompatible materials, orthopaedic prostheses, materials used in technical components, civil, mechanical, chemical engineering materials, etc.

The time-dependent material characteristics - the relaxation modulus $E(t)$ and the creep compliance modulus $C(t)$ are of great importance within rheology. They stand as an effective tool for the categorization of materials.

The state-of-the-art rheology under the isothermal and isobaric conditions reveals the time-dependent moduli participation in other phenomena, as a generalization to the multi-dimensional case, numerical approximation, and integral interpretation of the constitutive relations. Accordingly, the hereditary integrals as an inverse to the conditional stiffness, carry the information about the load and the mechanical response history. The time

invariant and time variant behaviour is focused. Since polymers are typically represented by the chain rheological model, the Prony series approximation is commonly employed with another viscoelastic phenomena - time retardation and time relaxation spectra are utilized. As rheology is a relatively young science, theoretical and applied investigation of viscoelastic models and rheological models, in general, are still in the focus of many researchers. The wide use of such investigations is undoubted.

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Nomenclature

A	action
$\mathbf{A}\,[Pa]$	anisotropic elasticity tensor
$\mathbf{A}_{dev}\,[Pa]$	deviatoric anisotropic elasticity tensor
$\mathbf{A}_{vol}\,[Pa]$	volumetric anisotropic stiffness tensor
(B)	Burgers model
$C(t)\,[Pa]$	compliane modulus
$E\,[Pa]$	Young elasticity modulus
$E(t)\,[Pa]$	relaxation modulus
$\hat{E}$	conditional stiffness
(gK)	generalized Kelvin model
(gM)	generalized Maxwell model
G	deviatoric modulus
$h_c\,[\frac{J}{s.kg}]$	flux coefficient of non-mechanical energy
$h(x)$	Heaviside function
$\mathbf{H}$	elastic modulus
$\overset{\infty}{\mathbf{H}}$	equilibrium modulus
$\hat{\mathbf{H}}$	residual modulus
$\mathbf{H}^*$	dynamical modulus
$\bar{\mathbf{H}}$	storage modulus
$\bar{\bar{\mathbf{H}}}$	loss modulus
$H_\alpha\,[J]$	nonmechanical energy
(H)	Hookean element
K	volumetric modulus
(K)	Kelvin - Voigt model
(M)	Maxwell modell
$\vec{n}$	outer normal vector
$N_D \in \langle 0, 1\rangle\,[-]$	Deborah number
(N)	Newtonian element
(PTh)	Poynting - Thompson model
$\vec{q}$	heat flow vector

$Q\,[J]$	heat
$\vec{r}\,[m]$	position vector
(rK)	recurrent Kelvin model
(rV)	recurrent Voight model
R	reaction
$S\,[\frac{J}{K}]$	global entropy of the body
(S)	Schofield model
$t\,[s]$	time
$T\,[K]$	temperature
$U\,[\frac{J}{m^3.s}]$	deformation energy per volume and time unit
$U_d\,[\frac{J}{m^3.s}]$	dissipated energy per volume and time unit
$U_p\,[\frac{J}{m^3.s}]$	potential energy per volume and time unit
$W\,[\frac{J}{m^3.s}]$	work of outer forces per volume and time unit
(Z)	Zener model
$\varepsilon(t), \boldsymbol{\varepsilon}\,[-]$	strain, strain function in time, strain tensor
$\zeta\,[\frac{m^2}{s}]$	dynamic viscosity coefficient
$\eta\,[Pa.s]$	dynamic viscosity coefficient
$\lambda\,[-]$	Lamé coefficient
$\lambda_c\,[\frac{1}{s}]$	characteristic equation root within creep test
$\lambda_r\,[\frac{1}{s}]$	characteristic equation root within relaxation test
$\mu\,[-]$	Poisson ratio
$\rho\,[\frac{kg}{m^3}]$	density
$\sigma(t), \boldsymbol{\sigma}\,[Pa]$	stress, stress function in time, stress tensor
$\boldsymbol{\sigma}_{dev}$	deviatoric viscosity tensor
$\boldsymbol{\sigma}_{vol}$	volumetric viscosity tensor
$\tau\,[Pa]$	shear stress in Chapter 2.1.2
$\tau\,[s]$	time; after Chapter 2.1.2

Index

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Mária Minárová graduated from Faculty of Mathematics, Physics and Informatics of Comenius University in Bratislava in study branch Mathematical analysis. Within the last two years of her study, she was simultaneously involved in the computer graphic study branch, resulting in her master thesis in this field. From the very beginning of her professional life, she teaches at Slovak Technical University in Bratislava. Upon several national or international projects she does the interdisciplinary and multidisciplinary research, concerning mostly mathematical modelling focused on applied mathematics in relations with various science branches as building physics, biomechanics, rheology, social sciences, medicine; computer implementation involved. The PhD thesis of her focuses on the mathematical modelling of moisture problems and moulds occurrence in buildings. The habilitation thesis deals with rheology. She acts as a reviewer for journals, as a jury member in national and European contests of young scientists and as an evaluator of national and European projects.

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